Viewing hybrid systems as products of control systems and automata

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1. Introduction

The purpose of this note is to show how hybrid systems may be modeled as products of nonlinear control systems and finite state automata. By a hybrid system, we mean a network of consisting of continuous, nonlinear control system connected to a discrete finite state automaton. Our point of view is that the automaton switches between the control systems, and that this switching is a function of the discrete input symbols or letters that it receives. There are several ways in which this may be modeled. The approach we take is a simple one: we show how a nonlinear control system may be viewed as a pair consisting of a bialgebra of operators coding the dynamics, and an algebra of observations coding the state space. We also show that a finite automaton has a similar representation. A hybrid system is then modeled by taking suitable products of the bialgebras coding the dynamics and the observation algebras coding the state spaces. An important advantage of viewing hybrid systems in an operator representation is that it is easy to specify algebraically how the various components of the hybrid system are "glued" together. One can then pass from the operator representation back to the state space representation when it is convenient. We simply outline the ideas in this note: for more details and examples, see [3].

In Section 2, we show how nonlinear control systems may be modeled in this way. In Section 3, we do the same for finite state automata. In Section 4, we define a suitable product of such systems, which we identify with a hybrid system. Section 5 contains some discussion and examples.

2. Operator representations of nonlinear control systems

In this section, we describe how to pass from the state space representation of a nonlinear control system to the operator representation. Let X denote the state space, let E_1 and E_2 denote vector fields, and let $t \to u_j(t)$ denote controls. Then the dynamics are described by

$$\begin{aligned} \dot{x}(t) &= u_1(t)E_1(x(t)) + u_2(t)E_2(x(t)), \\ x(0) &= x^0 \in X. \end{aligned}$$

Let $k = \mathbf{R}$ denote the real numbers, and let X denote the state space of the system. Define the *observation algebra* R to be the commutative algebra of all functions

$$R = \{ f : X \longrightarrow k \}.$$

If X carries an additional structure (such as being smooth or algebraic), then we require the same of the maps f. We also define a map (the *augmentation*)

$$\epsilon : R \longrightarrow k, \quad f \mapsto f(x^0).$$

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Let $H = k<E_1, E_2>$ denote the free associative algebra generated by the *symbols* E_1 and E_2. Here we are overloading E_j, so that it denotes either a vector field or a symbol depending upon the context. We call H the *dynamical algebra* associated with the control system. There is a natural action of H, given by the action of differential operators on observation functions. The algebraic structure of H is very rich: H is a bialgebra, so that the dual of H also carries an algebraic structure. This bialgebra structure interacts nicely with R, making R an H-module algebra. See [2] for further details.

To summarize, given a nonlinear control system in its state space representation, this process yields a pair (H, R), which we call the *operator representation* of the control system.

3. Operator representations of finite state automata

In this section, we describe how to pass from the state space representation of a finite state automaton to the operator representation. Let k be a field of characteristic 0. Let Ω denote the input symbols and $W = \Omega^*$ denote the free semigroup consisting of the words generated by Ω. Let

$$S = \{s^0, s^1, \ldots, s^n\}$$

denote the state space, and s^0 denote the intital state. The dynamics are described by specifying a transition for each input symbol α and each state s^j

$$S \times \Omega \longrightarrow S, \quad (s^j, \alpha) \mapsto s^j \cdot \alpha.$$

As usual, this action extends to an action of the words on the states

$$S \times W \longrightarrow S, \quad (s, w) \mapsto s \cdot w.$$

Define the observation algebra R and augmentation as before:

$$R = \{\, f : S \longrightarrow k \,\}$$

$$\epsilon : R \longrightarrow k, \quad f \mapsto f(s^0).$$

The dynamical algebra H is the semigroup algebra kW, defined to be the algebra over k consisting of finite formal sums of words. Again there is a natural action of the dynamical algebra H on the observation R,

$$L_w(f)(s) = f(s \cdot w), \quad w \in W, \quad s \in S, \quad f \in R.$$

Here L_w denotes a left action of W on R.

To summarize, given a finite state automaton, this process yields a pair (H, R), which we call the *operator representation* of the automaton.

4. Hybrid Systems

The basic idea is that a hybrid system is constructed out of components consisting of nonlinear control systems and discrete automata by taking suitable products of the components in their operator representations. Assume now that we have a finite state automaton, as described in the section above, with alphabet Ω and states $s \in S$, and that these states parametrize operator representations (H_s, R_s) of nonlinear control systems, one for each state s. For simplicity, assume that

$$H_s \cong H_0 = k<\xi_1, \ldots, \xi_M>, \quad \text{all } s \in S,$$

where the ξ_i are formal symbols. This simply means that the dynamics of each control system are driven by the same number of vector fields—the control systems themselves may be different, since this is coded by the action of the H_i on the R_i.

The hybrid system has an operator representation given by

$$H = k\Omega^* \amalg H_0, \quad R = \bigoplus_{s \in S} R_s,$$

where $\amalg$ denotes the free product of the indicated associative algebras.

5. Remarks

In this section, we make some general remarks about this construction. First, observe that in the case that the hybrid system contains no automaton component ($\Omega = \emptyset$ and $S = \{s_0\}$), then the hybrid system reduces to a nonlinear control system, as described in [2]. Second, observe that in the case in which the hybrid system contains no nonlinear system components

(H_s and R_s are both the trivial algebra k, for all $s \in S$) then the hybrid system reduces to a finite state automaton, as described in [3]. Finally, it is not hard to extend the realization theorem proved in [2] to hybrid systems of the form described here. This theorem reduces in the two special cases just described to the Fliess nonlinear realization theorem [1] and the Myhill-Nerode theorem respectively [4].

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