An Algebraic Approach to Hybrid Systems

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Abstract

We propose an algebraic model for hybrid systems and illustrate its usefulness by proving theorems in realization theory using this viewpoint. By a hybrid system, we mean a collection of continuous nonlinear control systems associated with a discrete finite state automaton. The automaton switches between the continuous control systems, and this switching is a function of the discrete input symbols that it receives.

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1 Introduction

In this paper, we propose an algebraic formalism for hybrid systems and illustrate its usefulness by using it to prove some theorems about realization. Informally, by a hybrid system we mean a collection of continuous non linear control systems connected to a discrete finite state automaton. The automaton switches between the continuous control systems as a function of the discrete input symbols that it receives. Formal definitions are given below.

Our viewpoint for hybrid systems was strongly influenced by conversations with George Meyer, Anil Nerode, and Wolf Kohn. The field of hybrid systems is in a rapid state of development, with many different viewpoints under exploration. The collection [5] contains articles exploring several of these. In particular, although the questions considered are different, and the techniques used are quite different, we note that the models for hybrid systems used by Kohn and Nerode [15] are broadly similar to the those considered here [10].

The fundamental questions about control systems include their controllability, observability, and realization [13] and [12]. By realization, we mean the question: given an input-output behavior of a system, to find a dynamical system with inputs and outputs, which can produce the given input-output behavior. A common way to specify the input-output behavior of the system as a whole is use a formal series, or *generating series* [12]. The realization question then becomes: given a generating series of the appropriate type, to find a dynamical system with inputs and outputs whose associated generated series is the one specified. A fundamental result is a theorem of Fliess [3]. We note that the Myhill–Nerode Theorem in automata theory [11] can be also interpreted as a realization theorem. One of the goals of this paper is to study the realization problem for hybrid systems and to show how both the Myhill–Nerode Thereom and the Fliess Theorem are special cases of a more general theorem. Throughout this paper, we consider only the formal algebraic aspects of realization theory.

Our starting place is the simple observation that algebraically, a nonlinear control system may be viewed as a pair consisting of an algebra of operators coding the dynamics, and an algebra of observation functions coding the state space. A finite automaton has a similar representation. The dynamics of the hybrid system formed from the nonlinear control system and the automaton is determined by the free product of the corresponding algebra of operators

Figure 1: A hybrid system, from the point of view considered in this paper, consists of an automaton which accepts a discrete input symbol, makes a state transition, and outputs a discrete output symbol. The discrete output symbol then selects a continuous nonlinear control system. The nonlinear control system flows for a period of time and the cycle repeats.

Various alternatives are possible: the output symbol of the automaton can be used to select the nonlinear control system, the nonlinear control system together with a input, or the the nonlinear control system, together with an input and a time interval for the flow. A final alternative is to use the output of the nonlinear control system to select the next input symbol for the automaton.

of its two components. This product algebra then acts on an observation algebra which is the direct sum of the observation algebras of the component systems. An important advantage of this approach is that it is easy to specify algebraically how the various components of the hybrid system are joined.

We use bialgebras to model the dynamics of hybrid systems: this turns out to be natural for two reasons:

1. The dynamics of a nonlinear control system can be viewed algebraically as an algebra of higher order derivations; similarly, the dynamics of an automaton can be viewed as an algebra of endomorphisms. A bialgebra is an algebra which enjoys a number of natural actions on other algebras: in one extreme it can act as an algebra of endomorphisms; in another extreme as an algebra of higher order derivations.

2. The observation functions of a system are dual to the points in the state space in a precise sense. This duality is fundamental to our approach and is closely related to the fact that the dual of a bialgebra is also an algebra.

Related Work. There are a variety of interpretations for hybrid systems that involve viewing hybrid systems as nonlinear control systems connected to automata, in addition to the interpretation in this paper that use the automata to provide a generalized type of mode switching. Closely connected to this viewpoint is to use the automata to construct control laws for the underlying continous systems [14]. Alternatively, the automata may be used to select trajectories or collections of trajectories of the continous systems in order to satisfy performance specifications [15].

Hybrid systems have a variety of representations. In the state space representation, the states, inputs, and outputs of each component continuous system and automaton are described together with the input-output connections between the various systems. In the input-output representation, the inputs and outputs of the hybrid system as a whole are described.

In this paper, we use a different representation—the observation space representation. Roughly speaking, this may be viewed as dual to the state space representation. This representation forms the basis for the Heisenberg picture in quantum mechanics [4]; has been used to define discrete time control systems by Sontag [16]; and has been applied to the study of continuous time control systems by Bartosiewicz [1] and [2].

A brief annoucement of some of the ideas in this paper appeared in [7]. The viewpoint in this paper has been used to study flows of hybrid systems in [8]. Simulation software for hybrid systems modeled from this viewpoint is described in [9].

Organization of the paper. In Section 2, we define the bialgebra representations of a nonlinear control system and of a finite state automaton. In Section 3, we give a "low brow" definition of a hybrid system from this point of view. This definition is close to the "operational" definition arising when modeling a system of discrete and continuous components. In Section 4, we give a more abstract and algebraic definition of a hybrid system. This definition (in the special case of primitively generated bialgebras) has already been used by us in [6] to prove a realization theorem for nonlinear control systems and other more combinatorial systems, which generalizes the formal part of the Fliess realization theorem [3]. Finally, in Section 5, we give three examples of hybrid systems: a nonlinear continuous control system; an automaton, which can be viewed as a control system which "accepts" or "rejects" based on the sequence of input symbols (the control) and so defines the language of accepted words; and a simple example of a hybrid system consisting of two non linear control systems and a two-state automaton switching between them. The systems are chosen such that control (when in one state of the automaton) is always in the north-south direction, and control (when in the other state of the automaton) is always in the east-west direction.

In this section we also state and prove an analogy of the generalized Fliess realization theorem for finite state automata, which is a generalization of the Myhill–Nerode theorem.

The goal of this paper is to show how bialgebras provide a useful tool for modeling hybrid systems consisting of networks of continuous components, and to give a few simple examples. For this reason, we have not tried to give all the necessary algebraic background, but have given only the essential ideas. For more details on using bialgebras to study control systems, see [6]. For background on bialgebras, see [17].

Notation. Throughout this paper, k will be a field of characteristic 0, such as $\mathbf{R}$ or $\mathbf{C}$.

2 The observation space representation

We first describe the approach to control systems taken in [3]. A nonlinear control system is described by a differential equation of the form

$$\dot{x}(t) = \sum_{\mu=1}^{m} u_\mu(t) E_\mu(x(t)), \quad x(0) = s_0 \in k^N,$$

where E_μ are vector fields defined in neighborhood of $x^0 \in k^N$, and the $t \mapsto u_\mu(t)$ are controls. Througout this paper, we restrict attention to control systems of this type, in which the inputs enter linearly. Define the observation algebra to be set of all smooth functions f on the state space $S = k^N$

$$R = \{ f \mid S \longrightarrow k \}. \tag{1}$$

Note that

1) R is a commutative k-algebra.

2) The maximal ideals R are in one to one correspondence with the points in the state space; in particular, the map $\alpha : R \longrightarrow k$ defined by $\alpha(f) = f(s_0)$ defines a maximal ideal $\ker \alpha$, which can be identified with the initial point $s_0 \in S$.

3) The vector fields E_μ may be viewed as derivations of R.

Let H denote the free Lie algebra generated by E_1, …, E_M. Then H acts naturally on R, since the E_μ do. It turns out that H is not only an algebra, but is a bialgebra [17].

To summarize, given a nonlinear control system, we obtain the

Observation Space Representation.

1) a bialgebra H;

2) a commutative algebra R coding the state space, and a map α coding the initial condition;

3) an action of H on R coding the dynamics.

Given these data, it is easy to recover the state space by taking the maximal ideals of R. From this and the action of H on R, it is easy to recover the derivations E_μ by letting the E_μ act on a basis ξ_i which is dual to a basis x_i for the state space.

We next describe a similar construction works for finite automata. Let Ω denote a finite alphabet of input letters and let Ω^* denote the set of strings of letters of Ω. Then Ω^* is a semigroup with identity and $H = k\Omega^*$ is a semigroup algebra. An automaton is a finite set of states S, an initial state $s_0 \in S$, and a transition map $\delta : S \times \Omega \longrightarrow S$, which extends to a map

$$\delta : S \times \Omega^* \longrightarrow S.$$

Define the observation algebra R as in Equation (1). Note that properties (1) and (2) hold with (3) replaced by

3) the elements of Ω act on R as endomorphisms via

$$\omega \cdot f(s) = f(\delta(s,\omega)) = f(s \cdot \omega),$$

for $f \in R$, $\omega \in \Omega$. This action extends to an action of the semigroup algebra $H = k\Omega^*$ on R.

Once again it turns out that $H = k\Omega^*$ is a bialgebra [17], yielding an observation space representation of the automaton as above. Given a bialgebra representation, it is easy to recover the state space S from the maximal ideals of R, to recover the initial state from the map α, and to recover the transition map δ by testing the action of H on characteristic functions in R.

3 Hybrid systems

We now give a “low-brow” definition of a hybrid system. The working definition we use throughout the paper is more abstract and given in Section 4. A *hybrid system* on n nodes and m generators consists of the following data:

1) For each node i, a nonlinear control system given by the observation space represention (H_i, R_i). The underlying state spaces for the non linear systems is often the same, that is, all the R_i.

2) An automaton with n states with input symbols from the alphabet Ω.

3) For each $w \in \Omega^*$, an algebra endomorphism

$$T_w : R \longrightarrow R,$$

where $R = R_1 \oplus \cdots \oplus R_n$. The assignment of endomorphisms satisfies $T_{wz} = T_w T_z$.

The idea is that the algebra endomorphisms T_w specify how the automaton switches between the nonlinear control systems.

We require the following technical conditions to hold:

Condition A. If $H_0 = k{<}\xi_1, \ldots, \xi_m{>}$ denotes the free associative algebra on m generators, then

$$H_i \cong H_0, \quad \text{for } i = 1, \ldots, n.$$

Condition B. There is an $R_0 = k[[X_1, \ldots, X_N]]$ such that

$$R_i \cong R_0, \quad \text{for } i = 1, \ldots, n.$$

Condition C. The action of H_i on R_i is given as follows:

$$\xi_{j_r} \cdots \xi_{j_1} \cdot f = E^i_{j_r} \cdots E^i_{j_1} \cdot f,$$

where $E^i_1, \ldots, E^i_m$ are the derivations of R_i defining the nonlinear system i.

Note that Condition C means that the each state i is associated with a nonlinear system whose dynamics are defined by the corresponding vector fields E_1^i, ..., E_m^i.

The *observation space representation of the hybrid system* is the pair (H, R), where

$$H = H_0 \amalg k\Omega^*, \qquad R = R_1 \oplus \cdots \oplus R_n.$$

Here $\amalg$ denotes the free product of associative algebras.

4 Hybrid systems revisited

In this section, in order to give a precise definition of a hybrid system, we need the notions of a bialgebra and of an H-module algebra.

An *algebra* A over the field k is a k-vector space A equipped with a multiplication $A \otimes A \longrightarrow A$ mapping $a \otimes b \mapsto ab$ and a unit $k \longrightarrow A$ mapping $1 \in k \mapsto 1 \in A$. The algebra is called *augmented* if there is an algebra homomorphism $A \longrightarrow k$. A *coalgebra* C over the field k is a k-vector space C equipped with a comultiplication $C \longrightarrow C \otimes C$ and a counit $C \longrightarrow k$. A *bialgebra* H over the field k is a k-vector space H which has both an algebra and a coalgebra structure such that the comultiplication and the counit maps are algebra homomorphisms, or equivalently, such that the multiplication and unit maps are coalgebra morphisms. Two examples of bialgebras are as follows. Let G be a semigroup with unit. Then the group algebra kG consisting of all formal finite linear combinations of elements of G with comultiplication defined by $g \mapsto g \otimes g$ and counit defined by $g \mapsto 1$ for $g \in G$, is a bialgebra. Let L be a Lie algebra. Then the universal enveloping algebra $U(L)$ with comultiplication defined by $x \mapsto 1 \otimes x + x \otimes 1$ and counit defined by $x \mapsto 0$ for $x \in L$, is a bialgebra.

Let H be a bialgebra. An algebra R is called a left H-module algebra in case R is a left H-module and

$$h \cdot (ab) = \sum_{(h)} (h_{(1)} \cdot a)(h_{(2)} \cdot b),$$

for all a, $b \in R$, $h \in H$. Here we write the comultiplication $H \longrightarrow H \otimes H$ as the map $h \mapsto \sum_{(h)} h_{(1)} \otimes h_{(2)}$. The algebra H^* has a left H-module algebra structure given by $(h \rightharpoonup f)(k) = f(kh)$, for $h,\ k \in H,\ f \in H^*$. The notion of a right H-module algebra is defined similarly.

We now give a second definition of a hybrid system. A *hybrid system* in the observation space representation is a pair (H, R) where

1) H is a bialgebra over the field k

2) R is a commutative algebra with augmentation $\alpha : R \longrightarrow k$.

3) there is an action of H on R making R into a left H-module algebra.

We leave it as an exercise to check that the low brow definition of a hybrid system yields a hybrid system as defined here. Properties (1) and (2) clearly hold. Property (3) holds because R is a H_0-module algebra and a $k\Omega^*$-module algebra, and H is the free product of H_0 and $k\Omega^*$.

Our main construction to obtain interesting examples of hybrid systems is as follows:

Main construction. Let G be a semigroup with identity, and let L be a Lie algebra over k. Let $H = kG \amalg U(L)$ be the free product of kG and $U(L)$. Then H is a bialgebra over k, with comultiplication and counit induced by those of kG and $U(L)$.

Suppose that R is an commutative algebra with unit and with augmentation α, that the elements of G act as algebra endomorphisms of R, and that the elements of L act as derivations of R. Then R is a kG-module algebra, and a $U(L)$-module algebra. It follows that R is a H-module algebra, as required for a hybrid system.

For specific applications, the semigroup G will be freely generated by a finite alphabet Ω, the Lie algebra L will be freely generated by a finite set $\{\xi_1, \ldots, \xi_M\}$, and the algebra R will be the direct sum of a finite number of algebras, each of which is a power-series algebra in finitely many variables.

If we wish to consider input-output systems, we also have an observation $f \in R$. In that case, we call $p \in H^*$ given by $p(h) = \alpha(f \cdot h)$, for $h \in H$, the *generating series* associated with the hybrid system (H, R, f). (See [6] for details.)

5 Some examples

This section contains three examples of the main construction: a nonlinear control system, an automaton, and a simple hybrid system.

5.1 Example — continuous systems

We first give the example of a continuous control system. In the main construction, let $G = \{\epsilon\}$, $R = k[[X_1, \ldots, X_N]]$, and L be the free Lie algebra generated by derivations $E_1, \ldots, E_M$ of R. Then $H \cong U(L)$, and we get a continuous control system as described in [6].

For these types of control systems, we have the following realization theorem. See [6] for a proof; but note that in this reference, we use a left action of H on R rather than a right action.

Theorem 1 (Fliess) *Let $p \in H^*$. Then the following are equivalent:*

1) $\dim(p \leftharpoonup L)$ *is finite;*

2) *there exists an commutative left H-module algebra R with augmentation α, such that* $\dim(\ker\alpha)/(\ker\alpha)^2$ *finite and $f \in R$ such that $p(h) = \alpha(h \cdot f)$ for all $h \in H$;*

3) *there exists an commutative left H-module algebra $R \subseteq H^*$ with $R \cong k[[X_1, \ldots, X_N]]$ with augmentation α, and $f \in R$ such that $p(h) = \alpha(h \rightharpoonup f)$ for all $h \in H$.*

5.2 Example — discrete systems

We next give the example of a finite automaton. Let Ω be a finite alphabet, and let Ω^* be the set of strings of letters of Ω. Then Ω^* is a semigroup with identity the empty string ϵ. Let M be a finite automaton accepting the language $L \subseteq \Omega^*$, let S be the set of states of the automaton, let $s_0 \in S$ be the initial state, and let $F \subseteq S$ be the set of accepting states, that is, $w \in \Omega^*$ is *accepted* by the automaton if and only if $s \cdot w \in F$. Let R be the algebra of k-valued functions on S. Then R is a commutative algebra with augmentation α given by $\alpha(r) = r(s_0)$. Let

$$f(s) = \begin{cases} 1 & \text{if } s \in F \\ 0 & \text{if } s \notin F. \end{cases}$$

The observation function $f \in R$ so defined is simply the characteristic function of the set of accepting states.

Note that $w \in L$ if and only if $s_0 \cdot w \in F$ if and only if $f(s_0 \cdot w) = 1$ if and only if $p(w) = \alpha(w \cdot f) = 1$. Therefore the generating series p in this case is the characteristic function of the language accepted by the automaton S. Also, note that this corresponds to the case $G = \Omega^*$ and $L = 0$ (hence $U(L) = k$) in our main construction.

We will prove the following realization theorem analogous to Theorem 1.

Theorem 2 (Myhill–Nerode) *Let G be a semigroup with unit, and let $H = kG$. Let $p \in H^*$. Then the following are equivalent:*

1) $\dim(H \rightharpoonup p)$ *is finite and there exists a non zero polynomial $Q(X) \in k[X]$ such that $Q(p) = 0$;*

2) $\dim(H \rightharpoonup p \leftharpoonup H)$ *is finite and there exists a non zero polynomial $Q(X) \in k[X]$ such that $Q(p) = 0$;*

3) $\dim(p \leftharpoonup H)$ *is finite and there exists a non zero polynomial $Q(X) \in k[X]$ such that $Q(p) = 0$;*

4) *there exists a finite dimensional commutative left H-module algebra R with augmentation α and $f \in R$ such that $p(h) = \alpha(h \cdot f)$ for all $h \in H$;*

5) *there exists an augmented commutative left H-module algebra $R \subseteq H^*$ with R isomorphic to the algebra of k-valued functions on some finite set S and $f \in R$ such that $p(h) = \alpha(h \rightharpoonup f)$ for all $h \in H$.*

Proof: It is immediate that (2) implies (1).

We prove that (3) implies (2). Let $I = \{ h \in H \mid (p \leftharpoonup H) \leftharpoonup h = 0 \}$. Then I is the maximal two sided ideal in H such that $p \leftharpoonup I = 0$. Since $p \leftharpoonup H$ is finite dimensional, and H/I is isomorphic to a subalgebra of $\mathrm{End}_k\, p \leftharpoonup H$, it follows that $\dim H/I$ is finite. Since I is a two sided ideal and $p(I) = p \leftharpoonup I(1) = 0$, it follows that $H \rightharpoonup p \leftharpoonup H \subseteq I^{\perp} = (H/I)^*$ is finite dimensional.

It is immediate that (5) implies (4).

We prove that (4) implies (3). We first show that $p \leftharpoondown H$ is finite dimensional. Let $r_1, \ldots, r_n$ be a basis for R. Then there exist $x_1, \ldots, x_n \in H^*$ such that

$$h \cdot f = \sum_{i=1}^{n} x_i(h) r_i, \qquad \text{for all } h \in H.$$

Now

$$\begin{aligned} (p \leftharpoondown l)(h) &= p(lh) \\ &= \alpha(lh \cdot f) \\ &= \alpha(l \cdot h \cdot f) \\ &= \sum_{i=1}^{n} x_i(h) \alpha(l \cdot r_i). \end{aligned}$$

Therefore $p \leftharpoondown H \subseteq \sum k x_i$ is finite dimensional.

We next show that there exists a polynomial $Q(X)$ such that $Q(p) = 0$. Since R is finite dimensional and $f \in R$, there exists a polynomial $Q(X)$ such that $Q(f) = 0$. Let $w \in \Omega$. Then $r \mapsto w \cdot r$ is an algebra endomorphism. Therefore $Q(w \cdot f) = w \cdot Q(f) = 0$. Since $p(w) = \alpha(w \cdot f)$ and $\alpha : R \to k$ is an algebra homomorphism, it follows that $Q(p(w)) = Q(\alpha(w \cdot f)) = \alpha(Q(w \cdot f)) = 0$. Since H^* can be identified with the algebra of k-valued functions on Ω^*, it follows that $Q(p) = 0$.

We finally prove that (1) implies (5). Since $g \mapsto (w \rightharpoonup g)$ is an algebra endomorphism of H^*, and since p satisfies $Q(p) = 0$, it follows that $Q(w \rightharpoonup p) = 0$. Therefore $H \rightharpoonup p$ is finite dimensional and spanned by algebraic elements, so generates a finite dimensional commutative algebra $R \subseteq H^*$ which is a left H-module algebra. Since H^* contains no non zero nilpotent elements, R contains no non zero nilpotent elements, and so is semisimple. Therefore R is a direct sum of finitely many field extensions of k. Since H^* is the direct product of copies of k, all of these field extensions must be k. Therefore R is isomorphic to the set of functions from S, the set of maximal ideals of R, to k. Let $f = p \in R$. Then it immediate that $p(h) = \alpha(f \rightharpoonup h)$ for all $h \in H$, where $\alpha(g) = g(1)$ for all $g \in H^*$. This completes the proof of the theorem.

We discuss briefly the connection between Theorem 2 and the Myhill–Nerode Theorem as it is usually stated (such as in Theorem 3.1 of [11]). There the function p is the characteristic function of the language L being considered, and takes only the values 0 and 1 on elements of Ω^*, and so always satisfies the polynomial $Q(X) = X^2 - X$. We will use this fact freely in the following discussion.

Condition (5) of Theorem 2 is equivalent to the assertion that the language L is accepted by a finite automaton. The set S of maximal ideals of R is the set of states of the automaton; Ω acts on S as follows: if $\omega \in \Omega$, since $r \mapsto \omega \cdot r$ is an algebra homomorphism, it induces a map $S \to S$ on the set of maximal ideals of R. The augmentation $\alpha : R \to k$ gives a maximal ideal which is the initial state. The function $f \in R$ is the characteristic function of some subset of S which is the set of accepting states.

We now consider Condition (1) of Theorem 2. Define an equivalence relation on Ω^* by $w \sim w'$ if and only if $q(w) = q(w')$ for all $q \in H \rightharpoonup p$. In other words, $w \sim w'$ if and only if $p(wz) = p(w'z)$ for all $z \in \Omega^*$, if and only if $wz \in L$ exactly when $w'z \in L$ for all $z \in \Omega^*$. The traditional Myhill–Nerode Theorem states that L is accepted by a finite automaton if and only if this equivalence relation has finite index. The subalgebra of H^* generated by $H \rightharpoonup p$, which is finite dimensional if and only if $H \rightharpoonup p$ is finite dimensional, is the algebra of all functions on the equivalence classes of this equivalence relation. Therefore Condition (1) of Theorem 2 is equivalent to the assertion that this equivalence relation has finite index.

The Myhill–Nerode Theorem is a realization theorem, in that it describes when a formal language is "realized" as the language recognized by a finite automaton.

5.3 Example — a simple hybrid system

The example presented here is the "taxicab-on-the-streets-of-Manhattan" example: we have continuous control, but at any given time, there is no north-south control (State 1), and all control is of east-west motion, or vice-versa (State 2). This example illustrates a general construction which takes nonlinear systems and an automaton and constructs a hybrid system. We call this the direct sum. More generally, one can construct hybrid systems from nonlinear control systems and automata in a variety of ways, using different products and constructions. The direct sum is one of the simplest. Given a hybrid system, under certain conditions, one can decompose it into nonlinear control systems and automata.

Let $R_i = k[X_1, X_2]$, $i = 1, 2$, and let $R = R_1 \oplus R_2$. We specify the action of $H_0 = k{<}\xi_1, \xi_2{>}$ on R by specifying its actions on R_i, $i = 1, 2$: on R_1

$$\begin{array}{lll} \xi_1 & \text{acts as} & E_1^1 = \partial/\partial X_1, \\ \xi_2 & \text{acts as} & E_2^1 = 0; \end{array}$$

on R_2

$$\begin{array}{lll} \xi_1 & \text{acts as} & E_1^2 = 0, \\ \xi_2 & \text{acts as} & E_2^2 = \partial/\partial X_2. \end{array}$$

The semigroup G is freely generated by $\Omega = \{a_1, a_2\}$. The action of Ω (and thus of G) on R is given by specifying its action on R_i, $i = 1, 2$. Its action on R_1 is given as follows: let $\rho_{12} : R_1 \to R_2$ be the isomorphism sending $X_1 \in R_1$ to $X_1 \in R_2$, and $X_2 \in R_1$ to $X_2 \in R_2$. Then, for $f \in R_1$,

$$T_{a_i}(f) = \begin{cases} f \oplus \rho_{12}(f) & \text{if } i = 1, \\ 0 & \text{otherwise.} \end{cases}$$

Its action on R_2 is defined similarly. Intuitively, a_1 maps all states into State 1, and a_2 maps all states into State 2. The action of Ω on R is the transpose of this map. The "typical" element $(u_1\xi_1 + u_2\xi_2)a_2(v_1\xi_1 + v_2\xi_2) \in H = k\Omega^* \amalg H_0$ (assuming that State 1 is the initial state) is to be interpreted as flowing along $v_1E_1^1 + v_2E_2^1$, making a transition to State 2, and then flowing along $u_1E_1^2 + u_2E_2^2$.

More generally, if $\Sigma_1, \ldots, \Sigma_n$ are continuous control systems with observation space representations (H_i, R_i) satisfying Conditions A, B, and C, and M is an automaton with n states $\{s_1, \ldots, s_n\}$, over the alphabet Ω, we denote the hybrid system

$$\coprod_M \Sigma_i = k\Omega^* \amalg H_0,$$

where $T_w : R \longrightarrow R$, $w \in \Omega^*$, $R = R_1 \oplus \cdots \oplus R_n$, is given by

$$T_w(f) = \sum_{j=1}^{n} \rho_w^{ij}(f)$$

for $f \in R_i$, where $\rho_w^{ij} : R_i \longrightarrow R_j$ is defined by

$$\rho_w^{ij} = \begin{cases} \text{id} & \text{if } w \cdot s_j = s_i, \\ 0 & \text{otherwise.} \end{cases}$$

(Recall that $R_i \cong R_0 \cong R_j$; this allows us to identify R_i with R_j.)

We call $\coprod_M \Sigma_i$ the *direct sum* of the control systems $\Sigma_1, \ldots, \Sigma_n$ with respect to the automaton M.

In a subsequent paper, we will prove a realization theorem for hybrid systems analogous to Theorems 1 and 2.

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