A simple construction of bialgebra deformations*

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This is a draft of an article which appeared in Contemporary Mathematics, Volume 134, American Mathematical Soceity, Providence, Rhode Island, 1992, pages 115–117.

Abstract

Let A denote a bialgebra over a field k and let $A_t = A[[t]]$ denote the ring of formal power series with coefficients in A. Assume that A is also isomorphic to a free, associative algebra over k. We give a simple construction which makes A_t a bialgebra deformation of A. In typical applications, A_t is neither commutative nor cocommutative. This construction yields bialgebra deformations associated with families of trees.

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Formal deformations of bialgebras and quantum groups have also been studied from related viewpoints by Drinfeld [1], Gerstenhaber [2], and Gerstenhaber and Schack [3]

*This is an extended abstract. The final detailed version of this paper has been submitted for publication elsewhere. 1991 MR Classification: 16W30, 16S30. Key Words: Hopf algebras, deformations, trees.

†This research is supported in part by NASA grant NAG2-513, NSF Grant DMS-8904740, and by the Laboratory for Advanced Computing at the University of Illinois at Chicago.

We begin by giving some definitions related to power series and the completed tensor product. Suppose that k is a field and A and B are k-algebras. We let $A_t = A[[t]]$ denote the ring of formal power series over A with its usual multiplication. Let $f : A \longrightarrow B_t$ be a k-linear map and write $f(a) = \sum_{n=0}^{\infty} c_f(a,n)t^n$ for $a \in A$. Define a k-linear map $\widehat{f} : A_t \longrightarrow B_t$ by

$$\widehat{f}(\mathbf{a}) = \sum_{n=0}^{\infty}(\sum_{i+j=n} c_f(a_i,j))t^n, \quad \text{for } \mathbf{a} = \sum_{n=0}^{\infty} a_n t^n \in A_t.$$

Observe that we can define a cateogy $(Alg_k)_t$, whose objects are A_t and whose morphisms are k-linear maps $f : A_t \longrightarrow B_t$ satisfying $f = \widehat{f|_A}$, where $f|_A$ is the restriction of f to A.

We define the *completed tensor product* $A_t \widehat{\otimes}_{k_t} B_t$ of A_t and B_t over k_t in this category to be $(A \otimes_k B)_t$. For $\mathbf{a} = \sum_{n=0}^{\infty} a_n t^n \in A_t$ and $\mathbf{b} = \sum_{n=0}^{\infty} b_n t^n \in B_t$ we let

$$\mathbf{a} \widehat{\otimes} \mathbf{b} = \sum_{n=0}^{\infty}(\sum_{i+j=n} a_i \otimes b_j)t^n.$$

In this category, we can also define a morphism of completed tensor products. See [6] for details.

The notions of algebra, coalgebra, bialgebra and Hopf algebra in the category $(Alg_k)_t$ are the same as those in the category of vector spaces over the field k except, of course, the structure maps are required to be morphisms. Let (A, m, η) be an algebra over k, where $m : A \otimes A \longrightarrow A$ is multiplication and $\eta : k \longrightarrow A$ defines the unity of A. Then $(A_t, \widehat{m}, \widehat{\eta})$ is an algebra in $(Alg_k)_t$. It is easy to see that $\widehat{m}(\mathbf{a} \widehat{\otimes} \mathbf{b}) = \mathbf{ab}$ for $\mathbf{a}, \mathbf{b} \in A_t$. A morphism $f : A_t \longrightarrow B_t$ is a morphism of algebras if and only $f|_A : A \longrightarrow B_t$ is a map of k-algebras.

Suppose that (A_t, Δ, ϵ) is a coalgebra in $(Alg_k)_t$. We say that $K \in A_t$ is grouplike if

$$\Delta(K) = K \widehat{\otimes} K \qquad \text{and} \qquad \epsilon(K) = 1.$$

We say that $\ell \in A_t$ is nearly primitive if

$$\Delta(\ell) = \ell \widehat{\otimes} K + H \widehat{\otimes} \ell$$

for some grouplike elements $K, H \in A_t$. If $K = H = 1$ then ℓ is said to be primitive. For an algebra A over a field k of characteristic 0 we let $\exp(at) = \sum_{n=0}^{\infty}(\frac{a^n}{n!})t^n \in A_t$.

Now we construct deformations of enveloping algebras over a field of characteristic 0 which are free as associative algebras on a space of primitives.

Theorem 1 *Suppose that V is a vector space over a field k of characteristic 0. Turn the tensor algebra (A, m, η) of V into a bialgebra $(A, m, \eta, \Delta, \epsilon)$ by defining $\Delta(\ell) = \ell \otimes 1 + 1 \otimes \ell$ and $\epsilon(\ell) = 0$ for $\ell \in V$. Let $p, q \in V$ and write V as a direct sum of subspaces $V = P \oplus P'$, where $P = span(p, q)$. Then there is bialgebra deformation $(A_t, \widehat{m}, \widehat{\eta}, \widehat{\mathbf{\Delta}}, \widehat{\epsilon})$ of $(A, m, \eta, \Delta, \epsilon)$ such that*

a) $\widehat{\mathbf{\Delta}}(\ell) = \ell \widehat{\otimes} 1 + 1 \widehat{\otimes} \ell$ *for* $\ell \in P$,

b) $K = exp(pt)$ *and* $H = exp(qt)$ *are grouplike elements of* $(A_t, \widehat{\mathbf{\Delta}}, \widehat{\epsilon})$, *and*

c) $\widehat{\mathbf{\Delta}}(\ell) = \ell \widehat{\otimes} K + H \widehat{\otimes} \ell$ *for* $\ell \in P'$.

We comment that $K = \exp(t\,p) = 1$ when $p = 0$. If $p \neq 0$ and $q = 0$, for example, then the deformation of the theorem is not cocommutative. If $dim(V) > 1$ then the free algebra A is not commutative. In this case the deformation of the theorem is not commutative. Finally, we comment that the Hopf algebras of trees described in [4] can be written as a free associative algebra over a certain basis of primitives so that the theorem applies.

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