Bialgebra deformations of certain universal enveloping algebras

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Abstract

Let A denote a bialgebra over a field k and let $A_t = A[[t]]$ denote the ring of formal power series with coefficients in A. Assume that A is a free algebra over k with a basis of primitives. We give a simple construction which makes A_t a bialgebra deformation of A. Usually A_t is neither commutative nor cocommutative. This construction yields deformations of bialgebras associated with families of trees.

1 Introduction

Let A denote a bialgebra over a field k and let $A_t = A[[t]]$ denote the ring of formal power series with coefficients in A. Assume that A is a free algebra over k with a basis of primitives of A. We give a simple construction which makes A_t a bialgebra deformation of A. In typical applications the deformation is neither commutative nor cocommutative. An important class of examples covered by this theorem is provided by Hopf algebras associated with certain families of trees, as described in [9]. An extended abstract describing our results has appeared elsewhere [12].

In this paper, we develop the algebraic machinery required for computations involving formal power series with coefficients from a bialgebra. A calculus of this type is required for applications, such as [5]. Section 2 contains preliminary material on completed tensor products and formal power

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series, viewed from a purely algebraic viewpoint. This section may be of independent interest. Section 3 contains a general construction for deforming certain enveloping algebras.

Formal deformations of bialgebras and quantum groups has also been studied from a related viewpoint by Gerstenhaber and Schack [7], [8].

The construction described in this paper arises naturally from the computations in [5] involving formal computations of differential operators and related computations involving trees. As motivation for the paper, we use the remainder of this section to briefly sketch some of these ideas.

Let R denote the polynomial algebra $k[x_1, \ldots, x_N]$, and consider the formal symbols F_j defined by

$$F_j = \sum_{\mu=1}^{N} a_j^\mu \frac{\partial}{\partial x_\mu}, \quad j = 1, \ldots, M$$

as first order differential operators with coefficients in R. Elements in the free associative algebra $A = k<F_1, \ldots, F_M>$ on $F_1, \ldots, F_M$ may then be interpreted as higher order differential operators generated by the $F_j's$.

Let $\mathcal{LT}$ denote the set of finite, rooted trees labeled with the symbols $\{F_1, \ldots, F_M\}$ and let $k\{\mathcal{LT}\}$ denote the k-vector space with basis $\mathcal{LT}$. In [9] a Hopf algebra structure is defined on $B = k\{\mathcal{LT}\}$. A Hopf algebra homomorphism

$$\phi : A \longrightarrow B$$

is constructed in [11] and it is shown that B, as well as A, measures R to itself. Using this structure, algorithms for manipulating differential operators symbolically are translated into assertions about labeled trees. This leads to algorithms which can be exponentially faster than naive ones.

The derivation of algorithms [5] for numerically integrating the flow of the nonlinear system

$$\dot{x}(t) = F(x(t)), \quad x(0) = x^0 \in \mathbf{R}^N$$

leads to computations in the algebras A_t and B_t. In particular the element

$$\exp tF = \sum_{i=0}^{\infty} \frac{t^i}{i!} F^i \in A_t$$

and its image $K = \phi(\exp tF) \in B_t$ turn out to be grouplike elements when A_t and B_t are given an appropriate coalgebra structure. Finding effecient numerical algorithms is equivalent to computing other grouplike elements with various desirable properties in these algebras.

The coproduct in B naturally lifts to a coproduct in B_t under which K is grouplike. Perturbing this coproduct with the grouplike $K \in B_t$ leads to a coproduct on B_t which is not cocommutative, and gives B_t the structure of a deformation of B.

The close connection between trees and differential operators was first noted by Cayley [3], [4], and has been rediscovered many times since then. The work of Butcher [1], [2] in his study of higher order Runge-Kutta algorithms also involves formal sums of trees and is closely connected to the work here.

2 Power series and the completed tensor product

In this section, we present some preliminary material concerning power series and completed tensor products. Although the material in this section is self-contained, the reader is left with the straightforward, but tedious, exercise of filling in the details.

Suppose that k is a field and A, B and C are k-algebras. We let $A_t = A[[t]]$ denote the ring of formal power series over A with its usual multiplication given by

$$ab = \sum_{n=0}^{\infty}(\sum_{i+j=n} a_i b_j)t^n, \quad \text{where } a = \sum_{n=0}^{\infty} a_n t^n, b = \sum_{n=0}^{\infty} b_n t^n \in A_t.$$

Observe that A_t is a k_t-module. Also note that A can be regarded as the k-subalgebra of A_t of power series with only constant term.

Let $f : A \longrightarrow B_t$ be a k-linear map and write $f(a) = \sum_{n=0}^{\infty} c_f(a, n)t^n$ for $a \in A$. Define a k-linear map $f^{\wedge} : A_t \longrightarrow B_t$,

$$f^{\wedge}(a) = \sum_{n=0}^{\infty}(\sum_{i+j=n} c_f(a_i, j))t^n, \quad \text{for } a = \sum_{n=0}^{\infty} a_n t^n \in A_t.$$

Now suppose that $g : B \longrightarrow C_t$ is k-linear. Then the formula

$$c_{g \bullet f}(a, n) = \sum_{i+j=n} c_g(c_f(a, i), j), \quad \text{for } a \in A \quad \text{and } n \geq 0$$

determines a "composite" $g \bullet f : A \longrightarrow C_t$.

Lemma 1 *Suppose that A, B and C are algebras over a field k, and let $f : A \longrightarrow B_t$ and $g : B \longrightarrow C_t$ be k-linear maps. Then:*

(a) *$f^\wedge$ is k_t-linear.*

(b) *$f^\wedge$ is a map of k_t-algebras if and only if f is a map of k-algebras.*

(c) *$g^\wedge \circ f^\wedge = (g \bullet f)^\wedge$.*

Let $f : A_t \longrightarrow B_t$ be a k-linear map. We say that f is a *morphism* if $f = (f^\vee)^\wedge$, where $f^\vee$ is the restriction of f to A. Morphisms are k_t-linear by part (a) of the lemma. Since the identity map of A_t is clearly a morphism, by the part (c) of the lemma morphisms and objects A_t form a category $(Alg_k)_t$.

We define the *completed tensor product* $A_t \widehat{\otimes}_{k_t} B_t$ of A_t and B_t over k_t in this category to be $(A \otimes_k B)_t$. The completed tensor product is t-adically complete and is a k_t-module since it is a power series ring. For $a = \sum_{n=0}^{\infty} a_n t^n \in A_t$ and $b = \sum_{n=0}^{\infty} b_n t^n \in B_t$ we let

$$a \widehat{\otimes} b = \sum_{n=0}^{\infty} (\sum_{i+j=n} a_i \otimes b_j) t^n.$$

Note that $a \widehat{\otimes} b = a \otimes b$ for $a, b \in A$ under the embedding of $A \otimes B$ in $A_t \widehat{\otimes}_{k_t} B_t$. It is not hard to see that

$$(a \widehat{\otimes} b)(a' \widehat{\otimes} b') = aa' \widehat{\otimes} bb' \tag{1}$$

for $a, a' \in A_t$ and $b, b' \in B_t$. Let $\imath : A_t \times B_t \longrightarrow A_t \widehat{\otimes}_{k_t} B_t$ be defined by $\imath(a, b) = a \widehat{\otimes} b$. Then for fixed $a \in A_t$ and $b \in B_t$ the maps $\imath(a, \ \ ) : B_t \longrightarrow A_t \widehat{\otimes}_{k_t} B_t$ and $\imath(\ \ , b) : A_t \longrightarrow A_t \widehat{\otimes}_{k_t} B_t$ are morphisms. The pair $(\imath, A_t \widehat{\otimes}_{k_t} B_t)$ satisfies the following universal property: If $f : A_t \times B_t \longrightarrow A_t \widehat{\otimes}_{k_t} B_t$ has the property that $f(a, \ \ )$ and $f(\ \ , b)$ are morphisms for all $a \in A_t$ and $b \in B_t$, then there is a unique morphism $F : A_t \widehat{\otimes}_{k_t} B_t \longrightarrow C_t$ such that $F \circ \imath = f$. By virtue of the universal property of the completed tensor product there are (unique) isomophisms $(A_t \widehat{\otimes}_{k_t} B_t) \widehat{\otimes}_{k_t} C_t \simeq A_t \widehat{\otimes}_{k_t} (B_t \widehat{\otimes}_{k_t} C_t)$ and $A_t \widehat{\otimes}_{k_t} B_t \simeq B_t \widehat{\otimes}_{k_t} A_t$ such that $(a \widehat{\otimes} b) \widehat{\otimes} c \mapsto a \widehat{\otimes} (b \widehat{\otimes} c)$ and $a \widehat{\otimes} b \mapsto b \widehat{\otimes} a$ respectively. We will use the identifications $A_t \widehat{\otimes}_{k_t} k_t \simeq A_t$ and $k_t \widehat{\otimes}_{k_t} A_t \simeq A_t$ given by $a \widehat{\otimes} \alpha \mapsto \alpha a$ and $\alpha \widehat{\otimes} a \mapsto \alpha a$ respectively.

Suppose that $f : A_t \longrightarrow A'_t$ and $g : B_t \longrightarrow B'_t$ are morphisms. We define a morphism of completed tensor products $f \widehat{\otimes} g : A_t \widehat{\otimes}_{k_t} B_t \longrightarrow A'_t \widehat{\otimes}_{k_t} B'_t$ by setting $(f\widehat{\otimes}g)^\vee(a\otimes b) = f(a)\widehat{\otimes}g(b)$ for $a \in A$ and $b \in B$. The usual formalism for the linear tensor product of maps translates to

$$(f\widehat{\otimes}g)(a\widehat{\otimes}b) = f(a)\widehat{\otimes}g(b) \tag{2}$$

for $a \in A_t$ and $b \in B_t$ in this category.

By part (b) of Lemma 1 and (2) we have

Lemma 2 *Suppose that $f : A_t \longrightarrow A'_t$ and $g : B_t \longrightarrow B'_t$ are morphisms. If f and g are algebra maps then the morphism $f\widehat{\otimes}g : A_t\widehat{\otimes}_{k_t}B_t \longrightarrow A'_t\widehat{\otimes}_{k_t}B'_t$ is an algebra map.*

The tensor product of polynomial algebras $A[t]\otimes_k A[t]$ and the completed tensor product $A_t\widehat{\otimes}_{k_t}A_t$ are related by the algebra homomorphism $\pi : A[t]\otimes_k A[t] \longrightarrow (A\otimes A)[t]$ defined by $\pi((\sum_n a_nt^n)\otimes_k(\sum_n b_nt^n)) = \sum_n(\sum_{i+j=n} a_i\otimes b_j)t^n$ for $\sum_n a_nt^n, \sum_n b_nt^n \in A[t]$. Since the kernel of π is the ideal generated by the differences $a\otimes bt - at\otimes b$ for $a, b \in A$, we may think of π as t-linearization. Now let $a = \sum_{n=0}^\infty a_nt^n, b = \sum_{n=0}^\infty b_nt^n \in A_t$. For $n \geq 0$ set $a_{(n)} = \sum_{i=0}^n a_it^i$ and $b_{(n)} = \sum_{i=0}^n b_it^i$. Then $a\widehat{\otimes}b = lim_{n\to\infty}\pi(a_{(n)}\otimes b_{(n)})$; in other words $a\widehat{\otimes}b$ is the limit of the tensor product of the approximations $a_{(n)}, b_{(n)} \in A[t]$ to a, b respectively, after t-linearization by π.

The notions of algebra, coalgebra, bialgebra and Hopf algebra in the category $(Alg_k)_t$ are the same as those in the category of vector spaces over the field k except, of course, the structure maps are required to be morphisms. Let (A, m, η) be an algebra over k, where $m : A\otimes A \longrightarrow A$ is multiplication and $\eta : k \longrightarrow A$ defines the unity of A. Then $(A_t, \widehat{m}, \widehat{\eta})$ is an algebra in $(Alg_k)_t$. It is easy to see that $\widehat{m}(a\widehat{\otimes}b) = ab$ for $a, b \in A_t$. A morphism $f : A_t \longrightarrow B_t$ is a morphism of algebras if and only $f^\vee : A \longrightarrow B_t$ is a map of k-algebras. The definition of algebra map, coalgebra map, and bialgebra map for the corresponding objects in $(Alg_k)_t$ is the same as those in the category of vector spaces over k; again, except that the maps are required to be morphisms.

Our treatment of power series rings can be done more abstractly in terms of the convolution algebra. Let C be a coalgebra over k. We express the coproduct by $\Delta(c) = \sum c_{(1)}\otimes c_{(2)} \in C\otimes_k C$ for $c \in C$. Now let A, B be k-algebras. The space of linear maps $A_C = Hom_k(C, A)$ has a natural algebra structure, called the convolution algebra, which is described as follows. The product of $a, b \in A_C$ is defined by $a*b = \sum a(c_{(1)})b(c_{(2)})$ for $c \in C$. The unit of A_C is the map $\mathbf{1} \in A_C$ defined by $\mathbf{1}(c) = \epsilon(c)1$ for $c \in C$. Note that A may be regarded as a subalgebra of A_C under the identification $a(c) = \epsilon(c)a$ for $c \in C$.

We will call a k-linear map $f : A_C \longrightarrow B_C$ a morphism if

$$f(a)(c) = \sum f(a(c_{(1)}))(c_{(2)}) \tag{3}$$

for all $a \in A_C$ and $c \in C$. The calculation

$$\begin{aligned} \sum(a(c_{(1)}))(c_{(2)}) &= \sum a(c_{(1)})\epsilon(c_{(2)}) \\ &= \sum a(c_{(1)}\epsilon(c_{(2)})) \\ &= a(c) \end{aligned}$$

shows that the identity map of A_C is a morphism. Objects A_C and morphisms as defined in (3) form a category $(Alg_k)_C$.

Now let C be the coalgebra over k with linear basis $c_0, c_1, c_2, ...$ whose coalgebra structure is defined by

$$\Delta(c_n) = \sum_{i+j=n} c_i \otimes c_j \qquad \text{and} \qquad \epsilon(c_n) = \delta_{0,n} \qquad \text{for all } n \geq 0. \tag{4}$$

Thus the natural identification $A_C = A_t(a \mapsto \sum_{n=0}^{\infty} a(c_n)t^n)$ is an isomorphism of k-algebras. Morphisms correspond under the identification. Thus we may think of the categories $(Alg_k)_t$ and $(Alg_k)_C$ as the same.

Now let C be any coalgebra over k. A sequence of elements $c_0, c_1, c_2, ... \in C$ is called a sequence of divided powers if (4) holds. Probably the most basic example of a sequence of divided powers arises from a primitive element in a bialgebra when the characteristic of the ground field is 0. Suppose that A is a bialgebra over k and that $\ell \in A$ is primitive. Since Δ is multiplicative, we calculate by the binomial theorem that $\Delta(\ell^n) = (\Delta(\ell))^n = (\ell \otimes 1 + 1 \otimes \ell)^n = \sum_{i=0}^{n} \binom{n}{i} (\ell^{n-i} \otimes \ell^i)$ for $n \geq 0$. Thus when the characteristic of k is 0, it follows that $c_0, c_1, c_2, ...$ is a sequence of divided powers, where $c_n = \frac{\ell^n}{n!}$ for $n \geq 0$.

3 Deformations of certain enveloping algebras

In this section, we describe how to deform bialgebras which are free associative algebras. Applications are given in the next section.

The proof of the proposition below is really a matter of unravelling definitions.

Proposition 1 *Suppose that A is an algebra over a field k with a k-coalgebra structure (A, Δ, ϵ). Then $(A_t, \widehat{\Delta}, \widehat{\epsilon})$ is a coalgebra in $(Alg_k)_t$. Furthermore*

$$\widehat{\Delta}(a) = \sum_{n=0}^{\infty} (\Delta(a_n))t^n \qquad \textit{and} \qquad \widehat{\epsilon}(a) = \sum_{n=0}^{\infty} \epsilon(a_n)t^n$$

for $a = \sum_{n=0}^{\infty} a_n t^n \in A_t$.

Proof: Since $\Delta : A \longrightarrow A \otimes_k A \subseteq A_t \widehat{\otimes}_{k_t} A_t$ and $\epsilon : A_t \longrightarrow k \subseteq k_t$, we have that $c_{\widehat{\Delta}}(a, n) = (\Delta(a))\delta_{n,0}$ and $c_{\widehat{\epsilon}}(a, n) = \epsilon(a)\delta_{n,0}$ for $a \in A$ and $n \geq 0$. The formulas for $\widehat{\Delta}$ and $\widehat{\epsilon}$ follow from these observations.

It remains to show that $(A_t, \widehat{\Delta}, \widehat{\epsilon})$ is a coalgebra in $(Alg_k)_t$; that is

$$(\widehat{\Delta}\widehat{\otimes}I) \circ \widehat{\Delta} = (I\widehat{\otimes}\widehat{\Delta}) \circ \widehat{\Delta} \tag{5}$$

and

$$(\widehat{\epsilon}\widehat{\otimes}I) \circ \widehat{\Delta} = I = (I\widehat{\otimes}\widehat{\epsilon}) \circ \widehat{\Delta} \tag{6}$$

To establish (5), we need only show that the composites of morphisms $(\widehat{\Delta}\widehat{\otimes}I) \circ \widehat{\Delta}$ and $(I\widehat{\otimes}\widehat{\Delta}) \circ \widehat{\Delta}$ agree on A, or equivalently that $(\Delta \otimes I) \circ \Delta = (I \otimes \Delta) \circ \Delta$. Thus (5) is established. By the same type of argument (6) holds. This concludes our proof.

Suppose that (A_t, Δ, ϵ) is a coalgebra in $(Alg_k)_t$. We say that $K \in A_t$ is grouplike if

$$\Delta(K) = K\widehat{\otimes}K \qquad \text{and} \qquad \epsilon(K) = 1.$$

We say that $\ell \in A_t$ is nearly primitive if

$$\Delta(\ell) = \ell\widehat{\otimes}K + H\widehat{\otimes}\ell$$

for some grouplike elements $K, H \in A_t$. If $K = H = 1$ then ℓ is said to be primitive.

For an algebra A over a field k of characteristic 0 we let $exp(at) = \sum_{n=0}^{\infty}(\frac{a^n}{n!})t^n \in A_t$. The following corollary gives the relationship between sequences of divided powers and grouplike elements.

Corollary 1 *Suppose that A is an algebra over a field k which has a k-coagebra structure (A, Δ, ϵ). Let $(A_t, \widehat{\Delta}, \widehat{\epsilon})$ be the resulting coalgebra in $(Alg_k)_t$. Then:*

(a) *Let $K = \sum_{n=0}^{\infty} a_n t^n \in A_t$. Then K is grouplike if and only if $a_0, a_1, a_2, \ldots$ is a sequence of divided powers in A.*

(b) *Suppose that the characteristic of k is* 0*. If $a \in A$ is primitive, then $K = exp(at)$ is a grouplike element of A_t.*

Now we construct deformations of enveloping algebras over a field of characteristic 0 which are free as associative algebras on a subspace of primitives.

Theorem 1 *Suppose that V is a vector space over a field k of characteristic 0. Turn the tensor algebra (A, m, η) of V into a bialgebra $(A, m, \eta, \Delta, \epsilon)$ by defining $\Delta(\ell) = \ell \otimes 1 + 1 \otimes \ell$ and $\epsilon(\ell) = 0$ for $\ell \in V$. Let $p, q \in V$ and write V as a direct sum of subspaces $V = P \oplus P'$, where $P = span(p, q)$. Then there is bialgebra deformation $(A_t, \widehat{m}, \widehat{\eta}, \widehat{\boldsymbol{\Delta}}, \widehat{\epsilon})$ of $(A, m, \eta, \Delta, \epsilon)$ such that*

a) $\widehat{\boldsymbol{\Delta}}(\ell) = \ell \widehat{\otimes} 1 + 1 \widehat{\otimes} \ell$ *for* $\ell \in P$*,*

b) $K = exp(pt)$ *and* $H = exp(qt)$ *are grouplike elements of* $(A_t, \widehat{\boldsymbol{\Delta}}, \widehat{\epsilon})$*, and*

c) $\widehat{\boldsymbol{\Delta}}(\ell) = \ell \widehat{\otimes} K + H \widehat{\otimes} \ell$ *for* $\ell \in P'$*.*

Proof: Define an algebra map $\boldsymbol{\Delta} : A \longrightarrow A_t \widehat{\otimes}_{k_t} A_t$ by setting $\boldsymbol{\Delta}(\ell) = \ell \widehat{\otimes} 1 + 1 \widehat{\otimes} \ell$ for $\ell \in P$ and $\boldsymbol{\Delta}(\ell) = \ell \widehat{\otimes} K + H \widehat{\otimes} \ell$ for $\ell \in P'$. Since $\boldsymbol{\Delta}$ and ϵ are maps of k-algebras, it follows that $\widehat{\boldsymbol{\Delta}} : A_t \longrightarrow A_t \widehat{\otimes}_{k_t} A_t$ and $\widehat{\epsilon} : A_t \longrightarrow k_t$ are as well by part b) of Lemma 1. Therefore $\widehat{\boldsymbol{\Delta}}$ and $\widehat{\epsilon}$ are algebra morphisms.

Thus to show that $(A, \widehat{m}, \widehat{\eta}, \widehat{\boldsymbol{\Delta}}, \widehat{\epsilon})$ is a bialgebra in $(Alg_k)_t$, we need only show that $(A_t, \widehat{\boldsymbol{\Delta}}, \widehat{\epsilon})$ is a coalgebra. First we show that K and H are grouplike elements of A_t. Observe that $\widehat{\boldsymbol{\Delta}}|_{T(P)} = \widehat{\Delta}|_{T(P)}$. Therefore K and H are grouplike elements by Corollary 1. Now we show that (5) and (6) hold, with $\widehat{\boldsymbol{\Delta}}$ replacing $\widehat{\Delta}$. Since $\boldsymbol{\Delta}$ is an algebra map, it follows by Lemma 2 that $(\widehat{\boldsymbol{\Delta}} \widehat{\otimes} I) \circ \widehat{\boldsymbol{\Delta}}$ and $(I \widehat{\otimes} \widehat{\boldsymbol{\Delta}}) \circ \widehat{\boldsymbol{\Delta}}$ are algebra maps. Thus to show that (5) holds for $\widehat{\boldsymbol{\Delta}}$, we need only note that $(I \widehat{\otimes} \widehat{\boldsymbol{\Delta}}) \circ \widehat{\boldsymbol{\Delta}}(\ell) = (\widehat{\boldsymbol{\Delta}} \widehat{\otimes} I) \circ \widehat{\boldsymbol{\Delta}}(\ell)$ for $\ell \in P$ and $\ell \in P'$ by part b) of Lemma 1. Similarly one can show that (6) holds for $\widehat{\boldsymbol{\Delta}}$ and $\widehat{\epsilon}$. Therefore $(A_t, \widehat{m}, \widehat{\eta}, \widehat{\boldsymbol{\Delta}}, \widehat{\epsilon})$ is a bialgebra in the category $(Alg_k)_t$.

It remains to show that $(A_t, \widehat{m}, \widehat{\eta}, \widehat{\boldsymbol{\Delta}}, \widehat{\epsilon})$ is deformation of $(A, m, \eta, \Delta, \epsilon)$. Since $\widehat{m}(a \widehat{\otimes} b) = ab + 0t + ...$ for $a, b \in A$, we need only show that $\widehat{\boldsymbol{\Delta}}(a) = \Delta(a) + t(\ \ ...\ \ )$ for $a \in A$. Now there exist functions $\Delta_n : A_t \longrightarrow A \otimes A$ for $n \geq 0$ such that $\widehat{\boldsymbol{\Delta}}(a) = \sum_{n=0}^{\infty} \Delta_n(a) t^n$ for all $a \in A_t$. Since $\widehat{\boldsymbol{\Delta}}$ is multiplicative, it follows that Δ_0 is also. Since $exp(at) = 1 + at + ...$ for $a \in A$, we conclude that $\widehat{\boldsymbol{\Delta}}(\ell) = (\ell \otimes 1 + 1 \otimes \ell) + t(\ \ ...\ \ )$ for all $\ell \in V$. Therefore $\Delta_0(\ell) = \Delta(\ell)$ for all $\ell \in V$. Since V generates A as an algebra, it follows that $\Delta_0 = \Delta$. This concludes the proof.

We comment that $K = exp(t\,p) = 1$ when $p = 0$. When $p = q = 0$, the deformation described in the theorem is cocommutative and is a trivial deformation. Assume now that the subspace P' is not zero. When $p \neq 0$ and $q = 0$, the deformation is not cocommutative. Hence, there is no bialgebra isomorphism between these two structures, and thus the $p \neq 0$ and $q = 0$ case is not a trivial deformation. If $dim(V) > 1$ then the free algebra A is not commutative. In this case the deformation is not commutative.

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