Hopf Algebras of Heap Ordered Trees and Permutations

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November 2, 2007

Abstract

A standard heap ordered tree with $n+1$ nodes is a finite rooted tree in which all the nodes except the root are labeled with the natural numbers between 1 and n, and that satisfies the property that the labels of the children of a node are all larger than the label of the node. Denote the set of standard heap ordered trees with $n+1$ nodes by $\mathcal{T}_n$. Let

$$k\mathcal{T} = \bigoplus_{n \geq 0} k\mathcal{T}_n.$$

It is known that there are Hopf algebra structures on $k\mathcal{T}$. Let $\mathfrak{S}_n$ denote the symmetric group on n symbols. Let

$$k\mathfrak{S} = \bigoplus_{n \geq 0} k\mathfrak{S}_n.$$

We give a bialgebra structure on $k\mathfrak{S}$, and show that there is a natural bialgebra isomorphism from $k\mathcal{T}$ to $k\mathfrak{S}$.

1 Introduction

We denote the set of standard heap ordered trees on $n+1$ nodes by $\mathcal{T}_n$. Let $k\mathcal{T}_n$ denote the vector space over the field k whose basis is the set of trees in $\mathcal{T}_n$, and let

$$k\mathcal{T} = \bigoplus_{n \geq 0} k\mathcal{T}_n.$$

In [2], we defined a noncommutative product and a cocommutative coproduct on $k\mathcal{T}$ that make $k\mathcal{T}$ it a Hopf algebra. Another product is given in [2, Ex. 6.2] which gives another Hopf algebra structure on $k\mathcal{T}$.

Let $\mathfrak{S}_n$ denote the symmetric group on n symbols, let $k\mathfrak{S}_n$ denote the vector space over the field k with basis $\mathfrak{S}_n$, and let

$$k\mathfrak{S} = \bigoplus_{n\geq 0} k\mathfrak{S}_n.$$

In [1], Aguiar and Sottile showed that there is a filtration on the Malvenuto-Reutenauer Hopf algebra $k\mathfrak{S}$ such that the associated graded dual is isomorphic to $k\mathcal{T}$. In [4] Malvenuto and Reutenauer showed that there is a Hopf algebra structure on $k\mathfrak{S}$ that is related to the Solomon descent algebra [5]. The Hopf algebra structure they introduced on $k\mathfrak{S}$ is noncommutative, noncocommutative, self-dual, and graded.

In [3, Prop. 3.3] a Hopf algebra structure on $k\mathfrak{S}$ is given: the product is the shifted concatenation of permutations; the coalgebra structure is given by the fact that the primitive elements are freely generated as a Lie algebra by the connected permutations (cycles). It is proved that this Hopf algebra is isomorphic to the Hopf algebra of heap ordered trees using that $k\mathfrak{S}$ is isomorphic to $U(P(k\mathfrak{S}))$. This Hopf algebra corresponds to the Hopf algebra of heap ordered trees with the product $\odot$ described in [2, Cor. 6.4].

In this paper we give another Hopf algebra structure on $k\mathfrak{S}$ which closely mirrors the Hopf algebra structure on heap ordered trees, and which we can easily prove is isomorphic to that Hopf algebra.

2 Heap Ordered Trees

In this section, we define heap ordered trees and standard heap ordered trees. In the next section, we define a Hopf algebra structure on the vector space whose basis is the set of standard heap ordered trees.

Definition 1 *A standard heap ordered tree on $n+1$ nodes is a finite, rooted tree in which all nodes except the root are labeled with the numbers $\{1, 2, 3, \ldots, n\}$ so that:*

1. *each label i occurs precisely once in the tree;*
2. *if a node labeled i has children labeled $j_1, \ldots, j_k$, then $i < j_1, \ldots, i < j_k$.*

We denote the set of standard heap ordered trees on $n+1$ nodes by $\mathcal{T}_n$. Let $k\mathcal{T}_n$ be the vector space over the field k whose basis is the set of trees in $\mathcal{T}_n$, and let

$$k\mathcal{T} = \bigoplus_{n\geq 0} k\mathcal{T}_n.$$

Definition 2 *A heap ordered tree is a rooted tree in which every node (including the root) is given a different positive integer label such that condition (2) is satisfied.*

A heap ordered tree differs from a standard heap ordered tree in that the root is also labeled, and that the labels can be taken from a larger set of positive integers.

Our convention for drawing trees is that the root is at the top and children are drawn so that a child of a node is to the left of another child if its label is lower.

In the following sections we use the operation that relabels a heap ordered tree by redefining the labels based on the order they are in and adding the same positive integer to each.

Definition 3 *Let t be a heap ordered tree. Define* $\mathrm{st}(t,m)$ *(where m is a non-negative integer) as follows: let $L=(j_1,\ldots,j_k)$ be the ordered list of integers which occur as labels of nodes of t. The ordered labeled tree* $\mathrm{st}(t,m)$ *is the same as t except that the label j_p is replaced by $p+m$. Note that if the root of t is unlabeled,* $\mathrm{st}(t,0)$ *is a standard heap ordered tree, which we denote by* $\mathrm{st}(t)$.

Sometimes it is helpful to think of $\mathrm{st}(t,m)$ as a way of "normalizing" the heap ordered tree t.

3 A Hopf Algebra Structure on Standard Heap Ordered Trees

In this section, we define a Hopf algebra structure on standard heap ordered trees, that is, a bialgebra structure which is graded and connected, and so has an antipode.

Suppose that $t_1 \in \mathcal{T}_m$ and $t_2 \in \mathcal{T}_n$ are standard heap ordered trees. Let $s_1, \ldots, s_r$ be the children of the root of $\mathrm{st}(t_1, n)$. We use the operation of deleting the root of a tree to produce a forest of trees, which we denote

$$\mathrm{B}_-(\mathrm{st}(t_1, n)) = \{s_1, \ldots, s_r\}.$$

Note that the s_i have labeled roots. If t_2 has $n+1$ nodes (counting the root), there are $(n+1)^r$ ways to attach the r subtrees of t_1 which have $s_1, \ldots, s_r$ as roots to the tree t_2 by making each s_i the child of some node of t_2. We denote the set of copies of t_2 with the s_i attached by

$$\mathrm{A}^+(\{s_1, \ldots, s_r\}, t_2).$$

The product $t_1 t_2$ of the trees t_1 and t_2 is the sum of these $(n+1)^r$ trees. We summarize:

Definition 4 *The product of the two standard heap ordered trees $t_1 t_2$ is:*

$$t_1 t_2 = \sum \mathrm{A}^+ \left(\mathrm{B}_-(\mathrm{st}(t_1, n)), t_2\right) \in k\mathcal{T}.$$

We define the coalgebra structure of standard heap ordered trees as follows:

$$\begin{aligned} \Delta(t) &= \sum_{X \subseteq \mathrm{B}_-(t)} \mathrm{st}(\mathrm{A}^+(X; e)) \otimes \mathrm{st}(\mathrm{A}^+(\mathrm{B}_-(t) \backslash X; e)) \\ \epsilon(t) &= \begin{cases} 1 & \text{if } t \text{ is the tree whose only node is the root,} \\ 0 & \text{otherwise.} \end{cases} \end{aligned}$$

Here, if $X \subseteq Y$ are multisets, $Y \backslash X$ denotes the set theoretic difference.

In [2], we show that this product and coproduct make $k\mathcal{T}$ into a Hopf algebra.

4 A Bialgebra Structure for Permutations

In this section, we define a bialgebra structure on $k\mathfrak{S}$. We begin with some notation.

Let $(\sigma_1 \sigma_2 \cdots \sigma_k)$ denote the cycle in $\mathfrak{S}_n$ which sends σ_1 to σ_2, σ_2 to σ_3, ..., and σ_k to σ_1. Every permutation is a product of disjoint cycles. If

$(\sigma_1\sigma_2\cdots\sigma_k)$ is a cycle, then there is a string naturally associated with the cycle that we write $\sigma_1\sigma_2\cdots\sigma_k$.

Now let $\sigma = (s_1)\cdots(s_r) \in \mathfrak{S}_m$ and $\tau = (t_1)\cdots(t_\ell) \in \mathfrak{S}_n$ be two permutations each written as a product of disjoint cycles. We denote the corresponding strings as $s_i = m_{i1}\cdots m_{ip_i}$ and $t_j = n_{j1}\cdots n_{jq_j}$ respectively. We call the elements of $\{n_{11},\ldots,n_{\ell q_\ell}\}$ *attachment points* for the cycles (s_1), …, (s_r) on the permutation $\tau = (t_1)\cdots(t_\ell)$. We will also define $\circ$ to be the $(n+1)^{th}$ attachment point.

Also, if σ is a permutation on $\{1,\ldots,k\}$, let $\mathrm{st}(\sigma,m)$ be the permutation on $\{m+1,\ldots,m+k\}$ that sends $m+i$ to $m+\sigma(i)$.

The definition of the heap product is simpler if we introduce the standard order of a permutation, which is defined as follows:

Definition 5 *We say that a permutation $\sigma \in \mathfrak{S}_m$ that is expressed as a product of cycles $\sigma = (s_1)\cdots(s_r)$ is in standard order if the cycles $s_i = m_{i1}\cdots m_{ip_i}$ are written so that*

1. $m_{i1} < m_{i2}$, $m_{i1} < m_{i3}$, …, $m_{i1} < m_{ip_i}$
2. $m_{11} > m_{21} > m_{31}$, …, $m_{i-1,1} > m_{i1}$

In other words, a product of cycles $\sigma = (s_1)\cdots(s_r)$ is written in standard order if each cycle $(s_i) = (m_{i1}\cdots m_{ip_i})$ starts with its smallest entry, and if the cycles (s_1), …, (s_r) are ordered so that their starting entries are decreasing. A permutation can always be written in standard order since disjoint cycles commute, and since a single cycle is invariant under a cyclic permutation of its string.

We now define the *heap product* of two permutations. Given two permutations $\sigma \in \mathfrak{S}_m$ and $\tau \in \mathfrak{S}_n$, write them in standard order $\sigma = (s_1)\cdots(s_r)$ and $\tau = (t_1)\cdots(t_\ell)$, where the string $s_i = m_{i1}\cdots m_{ip_i}$ and the string $t_j = n_{j1}\cdots n_{jq_j}$,

Definition 6 *We define the heap product $\sigma \# \tau$ of $\sigma \in \mathfrak{S}_m$ and $\tau \in \mathfrak{S}_n$ as follows:*

1. *replace σ by* $\mathrm{st}(\sigma,n)$*:*
2. *form all terms of the following form: If (s_i) is one of the cycles in σ, attach the string s_i to any one of the $n+1$ attachment points of τ; if the attachment point is one of n_{11},…, $n_{\ell q_\ell}$, say n_{jk}, place the string*

$s_i = m_{i1} \cdots m_{ip_i}$ *to the right of* n_{jk}*; otherwise (if the attachment point is* $\circ$*) we multiply the term we are constructing by* (s_i)*;*

3. *The product* $\sigma \# \tau$ *is the sum of all the terms constructed in this way, taken over all the cycles in* σ *and over all attachment points.*

Note that there are $(n+1)^r$ terms in $\sigma \# \tau$.

Some examples will illustrate this.

Example 7 *Let* $\tau = (n_1 n_2 \cdots n_p) \in \mathfrak{S}_p$ *be a single cycle and also let* $\sigma = (m_1 m_2 m_3) \in \mathfrak{S}_3$ *be a single cycle. We assume that* $\{m_1, m_2, m_3\} = \{p+1, p+2, p+3\}$*. We compute the product* $\tau \# \sigma$ *as follows:*

$$\begin{aligned} \sigma \# \tau &= (n_1 m_1 m_2 m_3 n_2 \cdots n_p) + (n_1 n_2 m_1 m_2 m_3 \cdots n_p) + \cdots \\ &\quad + (n_1 n_2 \cdots n_p m_1 m_2 m_3) + (n_1 n_2 \cdots n_p)(m_1 m_2 m_3), \end{aligned}$$

giving $p+1$ *terms.*

Example 8 *Let* $\sigma = (m_1)(m_2)(m_3) \in \mathfrak{S}_3$ *be the product of three 1-cycles, and let* $\tau = (n_1 n_2 n_3) \in \mathfrak{S}_3$ *be a 3-cycle. We assume that* $\{m_1, m_2, m_3\} = \{4, 5, 6\}$*. Then*

$$\begin{aligned} \sigma \# \tau &= (n_1 m_1 m_2 m_3 n_2 n_3) + (n_1 m_1 m_2 n_2 m_3 n_3) + (n_1 m_1 m_2 n_2 n_3 m_3) \\ &\quad + (n_1 m_1 m_3 n_2 m_2 n_3) + (n_1 m_1 n_2 m_2 m_3 n_3) + (n_1 m_1 n_2 m_2 n_3 m_3) \\ &\quad + \cdots \\ &\quad + (n_1 m_3 n_2 n_3)(m_1)(m_2) + (n_1 n_2 m_3 n_3)(m_1)(m_2) \\ &\quad + (n_1 n_2 n_3 m_3)(m_1)(m_2) + (n_1 n_2 n_3)(m_1)(m_2)(m_3) \end{aligned}$$

giving $4^3 = 64$ *terms.*

We now describe the coalgebra structure of $k\mathfrak{S}$.

We define a function $\mathrm{st}(\pi)$ from permutations to permutations as follows: let $\pi = (s_1) \cdots (s_p) \in \mathfrak{S}_n$ and let let $L = \{\ell_1, \ldots, \ell_k\}$ be the labels (in order) which occur in the s_i. (If π fixes i, we include a 1-cycle (i) as a factor in π.) The permutation $\mathrm{st}(\pi)$ is the permutation in $\mathfrak{S}_k$ gotten by replacing ℓ_j with j in π. For example, if $\pi = (13)(4)(57) \in \mathfrak{S}_7$, then $\mathrm{st}(\pi) \in \mathfrak{S}_5$ equals $(12)(3)(45)$.

The coalgebra structure of $k\mathfrak{S}$ is defined as follows. let $\pi = (s_1)\cdots(s_k) \in \mathfrak{S}_n$, and let $C = \{(s_1), \ldots, (s_k)\}$. If $X \subseteq C$ let $\rho(X) = \mathrm{st}(\prod_{(s_i)\in C}(s_i))$. Note that if $\rho(X) \in \mathfrak{S}_k$, then $\rho(C\backslash X) \in \mathfrak{S}_{n-k}$. Define

$$\begin{aligned}\Delta(\pi) &= \sum_{X\subseteq C} \rho(X) \otimes \rho(C\backslash X)\\ \epsilon(\pi) &= \begin{cases} 1 & \text{if } \pi \text{ is the identity permutation in } \mathfrak{S}_0,\\ 0 & \text{otherwise.}\end{cases}\end{aligned}$$

In Section 5, we show that there is a bialgebra isomorphism between the Hopf algebra of standard heap ordered trees and the bialgebra of permutations.

5 From Standard Heap Ordered Trees to Permutations

In this section, we define a map φ from standard heap ordered trees $\mathcal{T}_n$ to permutations $\mathfrak{S}_n$ and show that this gives a bialgebra homomorphism from $k\mathcal{T}$ to $k\mathfrak{S}$. We show that φ is an isomorphism by giving its inverse.

Definition 9 *We define the map α from heap ordered trees to strings recursively. Let t be a heap ordered tree: if the root of t has label i and children t_1, ..., t_k (read from left to right), then $\alpha(t)$ is the string $i\alpha(t_k)\cdots\alpha(t_1)$. Note that if t (whose root is labeled with i) has no children, then $\alpha(t)$ is the string i.*

It is important to note that this definition relies on the convention defined in Section 2. With this convention, children of a node are arranged from left to right in *increasing* order.

Let t be a standard heap ordered tree, and let t_1, ..., t_k be the heap ordered trees which are the children of the root of t. Then $\varphi(t)$ is the permutation $(\alpha(t_1))\cdots(\alpha(t_k))$.

Definition 10 *We recursively define a map β from strings to heap ordered trees. We define β on strings whose elements are either numbers or heap ordered trees. In comparisons which involve heap ordered trees, we use the label of the root of the tree in the comparison.*

Let s be a string $n_1 \cdots n_k$ with $n_1 < n_2$, ..., $n_1 < n_k$. A valid *substring is a substring $n_i \cdots n_j$ with $n_i < n_{i+1}$, ..., $n_i < n_j$ and either $n_i > n_{j+1}$ or $j = k$ (that is, n_j is the last entry in s). A string always has a valid substring (which might be the whole string).*

The map β is defined as follows: let $n_i \cdots n_j$ be a valid substring. Replace this substring with the heap ordered tree with root labeled with n_i, and with children of the root n_k ($i < k \leq j$) either labeled with n_k (if n_k is a number) or the tree itself (if n_k is a heap ordered tree).

Now we can define φ^{-1}: Let $\pi = (s_1) \cdots (s_k)$ be a permutation in $\mathfrak{S}_n$. If some number i in $\{1, \ldots, n\}$ does not occur in any of the strings s_1, ..., s_k, replace π by $\pi(i)$ so that we may assume that every number in $\{1, \ldots, n\}$ occurs explicitly in π. We can assume that π is in standard order.

We construct a standard heap ordered tree $t = \varphi^{-1}(\pi)$ as follows: first construct the root and let $\beta(s_1)$, ..., $\beta(s_k)$ be the children of that unlabeled root.

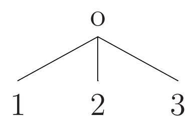
Theorem 11 *The map φ is a bialgebra morphism.*

PROOF: In forming the product in $k\mathcal{T}$ we attach the children of the root of the first multiplicand to nodes of the second multiplicand or to its root. In forming the product in $k\mathfrak{S}$ we attach cycles of the first multiplicand to attachment points of the second multiplicand. These are essentially the same operation.

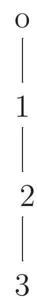
In the construction of the coproduct of $k\mathcal{T}$, for $t \in \mathcal{T}_n$ we use each subset of the set of children of the root on t to construct a new tree. In the construction of the coproduct on $k\mathfrak{S}$ for $\pi \in \mathfrak{S}_n$ we use each subset of the set of cycles of π to construct a new permutation. These are essentially the same operation.

Remark 12 *Since we already know that $k\mathcal{T}$ is a Hopf algebra with associative multiplication and that $k\mathfrak{S} \cong k\mathcal{T}$, this gives a proof that $k\mathfrak{S}$ is a Hopf algebra.*

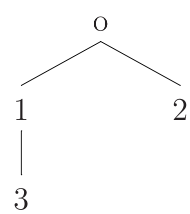
Here are some examples of φ:



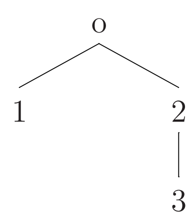
$\xrightarrow{\varphi}$ (1)(2)(3)



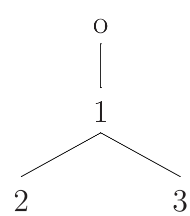
$\xrightarrow{\varphi}$ (123)



$\xrightarrow{\varphi}$ (13)(2)



$\xrightarrow{\varphi}$ (1)(23)



$\xrightarrow{\varphi}$ (132)

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