

Differential Algebra Structures on Families of Trees

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January 29, 2005

Abstract

It is known that the vector space spanned by labeled rooted trees forms a Hopf algebra. Let k be a field and let R be a commutative k -algebra. Let H denote the Hopf algebra of rooted trees labeled using derivations in $\text{Der}(R)$. In this paper, we introduce a construction that gives R a H -module algebra structure and show this induces a differential algebra structure of H acting on R . The work here extends the notion of a R/k -bialgebra introduced by Nichols and Weisfeiler.

This is a draft of a paper that will appear in *Advances in Applied Mathematics*.

1 Introduction

Let k be a field, R be a commutative k -algebra, and $\text{Der}(R)$ the Lie algebra of derivations of R . It is known that the vector space spanned by labeled rooted trees forms a Hopf algebra [4]. Let H denote the Hopf algebra of rooted trees whose non-root nodes are labeled using derivations $D \in \text{Der}(R)$ [4]. For such a Hopf algebra, we introduce a class of H -module algebras which we call Leibnitz, and give a construction which yields a variety of different Leibnitz H -module algebras (Theorem 3.11).

We also show how Leibnitz H -module algebras are related to Nichols and Weisfeiler's R/k -bialgebras [10], which arise in Hopf-algebra approaches to differential algebra (Theorem 4.7). In Section 5 we also give a method for describing quotients of Leibnitz H -module algebras (Theorem 5.8).

Hopf algebras can be used to simplify computations of derivations [5]. In the same way, Leibnitz H -module algebras can be used to simplify the symbolic computation of derivations acting on polynomials and other algebras of functions. This was used in [1] and [2] to derive geometrically stable numerical integration algorithms, although the results are presented differently.

The authors wish to thank the referee for various suggestions that improved the paper.

2 Bialgebras and trees

This section reviews some material from [4] on the Hopf algebra structure of trees. For background material on Hopf algebras see [14]. Throughout this paper k is a field.

Let R be a commutative k -algebra. Let \mathcal{D} be a vector space over k , and let \mathcal{S} be a subset of \mathcal{D} . Let $\mathcal{T}(\mathcal{S})$ denote the set of ordered trees in which each node other than the root is labeled with an element of \mathcal{S} . Let $k\{\mathcal{T}(\mathcal{S})\}$ be the vector space over k with basis the elements of $\mathcal{T}(\mathcal{S})$.

When we speak of a “subtree of a tree” we also include the (unlabeled) node in the tree to which the subtree is attached as the root of the subtree. When we refer to the “children of a node v ” we will sometimes mean the nodes which are attached to v as immediate descendants, and sometimes mean the full subtrees which are attached to v . Which sense is meant will be clear from context.

We have a grading on $k\{\mathcal{T}(\mathcal{S})\}$: $k\{\mathcal{T}(\mathcal{S})\}_r$ is spanned by the trees with

$r + 1$ nodes. There is a bialgebra structure on $k\{\mathcal{T}(\mathcal{S})\}$ given in [4] which we summarize here.

Multiplication in $k\{\mathcal{T}(\mathcal{S})\}$ is defined as follows. Let T_1 and T_2 be rooted trees. Remove the root from the tree T_1 to form a multiset \mathcal{F} of rooted trees. Let d be a function from \mathcal{F} to the set of nodes of T_2 . Let T_d be the rooted tree formed by adding an edge to link the root of $T' \in \mathcal{F}$ to the node $d(T')$ of T_2 . The order on the children of a node of T_d is given by declaring that the order of the children of a node of a tree in \mathcal{F} is preserved, that the order of the children of a node of T_2 is preserved, that if two roots of trees in \mathcal{F} are linked to the same node of T_2 , they are given the order they had as children of the root of T_1 , and that the root of T' precedes every child of the node $d(T')$ to which it is linked. Now

$$T_1 \cdot T_2 = \sum_d T_d,$$

where the sum ranges over all trees T_d formed as described above. This product is associative, and the tree with only one node is a multiplicative unit. See [4] for details and Figure 1 for some examples. (Note that in [4] multiplication is defined so that the root of $T' \in \mathcal{F}$ follows every child of the node $d(T')$; this will not change the application of the results we use from there.)

The coproduct in $k\{\mathcal{T}(\mathcal{S})\}$ is defined as follows. If T is a labeled ordered rooted tree, define

$$\Delta(T) = \sum_{\mathcal{X}} T_{\mathcal{X}} \otimes T_{\mathcal{F} \setminus \mathcal{X}},$$

where \mathcal{X} ranges over all sub-multisets of the ordered multiset \mathcal{F} described above. If \mathcal{Y} is a sub-multiset of \mathcal{F} , the labeled ordered tree $T_{\mathcal{Y}}$ is formed by adding edges to link the roots of the trees in \mathcal{Y} to a new root, preserving their original order and labels. In particular, $T_{\mathcal{F}} = T$ and $T_{\emptyset} = 1$ is the unit. The counit $\epsilon : k\{\mathcal{T}(\mathcal{S})\} \rightarrow k$ is defined as follows.

$$\epsilon(T) = \begin{cases} 1 & \text{if } T = 1 \\ 0 & \text{otherwise.} \end{cases}$$

In this construction, each non-root node retains its original label; recall that the root is not labeled. See [4] for more details and Figure 2 for some examples.

Since $k\{\mathcal{T}(\mathcal{S})\}$ is a graded bialgebra with $\dim k\{\mathcal{T}(\mathcal{S})\}_0 = 1$, it is a Hopf algebra.

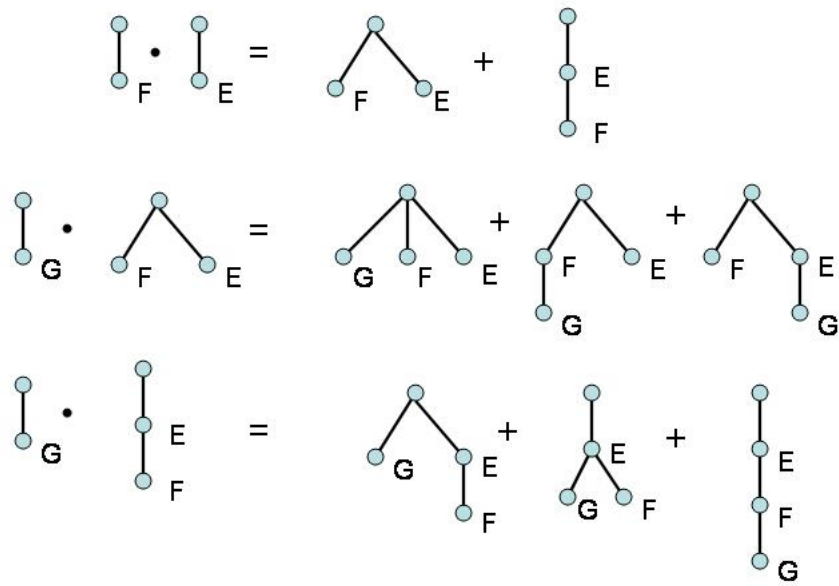


Figure 1: Some examples of multiplying labeled trees.

$$\begin{aligned}
\Delta \left(\begin{array}{c} \circ \\ | \\ \circ \\ \text{E} \end{array} \right) &= 1 \otimes \begin{array}{c} \circ \\ | \\ \circ \\ \text{E} \end{array} + \begin{array}{c} \circ \\ | \\ \circ \\ \text{E} \end{array} \otimes 1 \\
\Delta \left(\begin{array}{c} \circ \\ / \quad \backslash \\ \circ \quad \circ \\ \text{F} \quad \text{E} \end{array} \right) &= 1 \otimes \begin{array}{c} \circ \\ / \quad \backslash \\ \circ \quad \circ \\ \text{F} \quad \text{E} \end{array} + \begin{array}{c} \circ \\ | \\ \circ \\ \text{F} \end{array} \otimes \begin{array}{c} \circ \\ | \\ \circ \\ \text{E} \end{array} \\
&\quad + \begin{array}{c} \circ \\ / \quad \backslash \\ \circ \quad \circ \\ \text{F} \quad \text{E} \end{array} \otimes 1 + \begin{array}{c} \circ \\ | \\ \circ \\ \text{E} \end{array} \otimes \begin{array}{c} \circ \\ | \\ \circ \\ \text{F} \end{array}
\end{aligned}$$

Figure 2: Some examples of co-multiplying labeled trees.

We summarize this discussion in the following

Proposition 2.1 *Let k be a field and let \mathcal{S} be a vector space over k . Then $k\{\mathcal{T}(\mathcal{S})\}$ is a cocommutative graded connected Hopf algebra.*

3 H -module algebras

Let R be a commutative k -algebra, and let H be a k -bialgebra. Recall the following definition [8, 4.1.1]:

Definition 3.1 *The algebra R is a left H -module algebra if R is a left H -module for which*

$$h \cdot (rs) = \sum_{(h)} (h_{(1)} \cdot r)(h_{(2)} \cdot s),$$

where $h \in H$, $\Delta(h) = \sum_{(h)} h_{(1)} \otimes h_{(2)}$, and $r, s \in R$.

Example 3.2 Let $R = k[X_1, \dots, X_N]$. Then the Lie algebra \mathcal{D} of derivations of R has $\{\partial/\partial X_1, \dots, \partial/\partial X_N\}$ as an R -basis. Let H be the bialgebra $k\{\mathcal{T}(\mathcal{D})\}$ defined in Section 2. We define an H -module algebra structure on R as follows. Let $T \in k\{\mathcal{T}(\mathcal{D})\}$ be a labeled tree. Number the root of T with 0, and number the other nodes $1, \dots, m$. Let node i , $i > 0$, be labeled with

$$E_i = \sum_{\mu_i=1}^N r_{i\mu_i} \frac{\partial}{\partial X_{\mu_i}}, \quad (1)$$

where $r_{i\mu_i} \in R$. Suppose that node i , $i \geq 0$, has children j_1, \dots, j_k . Define

$$R(i; \mu_{j_k} \dots \mu_{j_1}) = \begin{cases} \frac{\partial}{\partial X_{\mu_{j_k}}} \cdots \frac{\partial}{\partial X_{\mu_{j_1}}} s & \text{if } i = 0, \\ \frac{\partial}{\partial X_{\mu_{j_k}}} \cdots \frac{\partial}{\partial X_{\mu_{j_1}}} r_{i\mu_i} & \text{otherwise.} \end{cases}$$

We will usually abbreviate $R(i; \mu_{j_k} \dots \mu_{j_1})$ by $R(i)$. Define

$$T \cdot s = \sum_{\mu_1, \dots, \mu_m=1}^N R(m) \cdots R(1)R(0),$$

for $s \in R$. It can be shown [6][Prop. 2] that this makes R into a left H -module algebra. We will prove the existence of more complex H -module algebra structures in Section 5.

We will generalize Example 3.2 in Proposition 3.9. Here is a specific case of the construction in Example 3.2.

Example 3.3 Consider the following two vector fields on \mathbf{R}^8 introduced in [3]:

$$\begin{aligned} E_1 &= \frac{\partial}{\partial x_1} \\ E_2 &= \frac{\partial}{\partial x_2} - x_1 \frac{\partial}{\partial x_3} + \frac{1}{2} x_1^2 \frac{\partial}{\partial x_4} + x_1 x_2 \frac{\partial}{\partial x_5} \\ &\quad - \frac{1}{6} x_1^3 \frac{\partial}{\partial x_6} - \frac{1}{2} x_1^2 x_2 \frac{\partial}{\partial x_7} - \frac{1}{2} x_1 x_2^2 \frac{\partial}{\partial x_8}. \end{aligned}$$

Then it is simple to check, for example, that

$$\begin{aligned} [E_2, E_1] &= \frac{\partial}{\partial x_3} - x_1 \frac{\partial}{\partial x_4} - x_2 \frac{\partial}{\partial x_5} + \frac{1}{2} x_1^2 \frac{\partial}{\partial x_6} + x_1 x_2 \frac{\partial}{\partial x_7} + \frac{1}{2} x_2^2 \frac{\partial}{\partial x_8}, \\ [[E_2, E_1], E_1] &= \frac{\partial}{\partial x_4} - x_1 \frac{\partial}{\partial x_6} - x_2 \frac{\partial}{\partial x_7}, \\ [[E_2, E_1], E_2] &= \frac{\partial}{\partial x_5} - x_1 \frac{\partial}{\partial x_7} - x_2 \frac{\partial}{\partial x_8}. \end{aligned}$$

See Figure 3 for the corresponding trees. Note also that:

$$[E_2, E_1](x_3) = 1, \quad [[E_2, E_1], E_1](x_4) = 1, \quad \text{and} \quad [[E_2, E_1], E_2](x_5) = 1.$$

We could continue this example by checking that the actions of the differential operators is the same as the actions of the trees.

We will use the following definition in the sequel.

Definition 3.4 Let \mathcal{D} be a vector space.

- a) Let $E \in \mathcal{D}$. Denote by $v(E)$ the labeled ordered tree with two nodes: the root, and a single child which is labeled with E .
- b) Let T_1, \dots, T_k be labeled ordered trees, and let $E \in \mathcal{D}$. Denote by $u(E; T_1, \dots, T_k)$ the labeled ordered tree whose root has one child, labeled with E , with which the roots of the subtrees T_1, \dots, T_k are identified. The ordering on the children of the node labeled with E in $u(E; T_1, \dots, T_k)$ is given by specifying that the children of the root of T_1 precede the children of the root of T_2, \dots , which precede the children of the root of T_k .

c) Let T_1, \dots, T_k be labeled ordered trees. Denote by $t(T_1, \dots, T_k)$ the labeled ordered tree formed by identifying the roots of the trees T_1, \dots, T_k . The ordering on the children of the root $t(T_1, \dots, T_k)$ is given by specifying that the children of the root of T_1 precede the children of the root of T_2, \dots , which precede the children of the root of T_k .

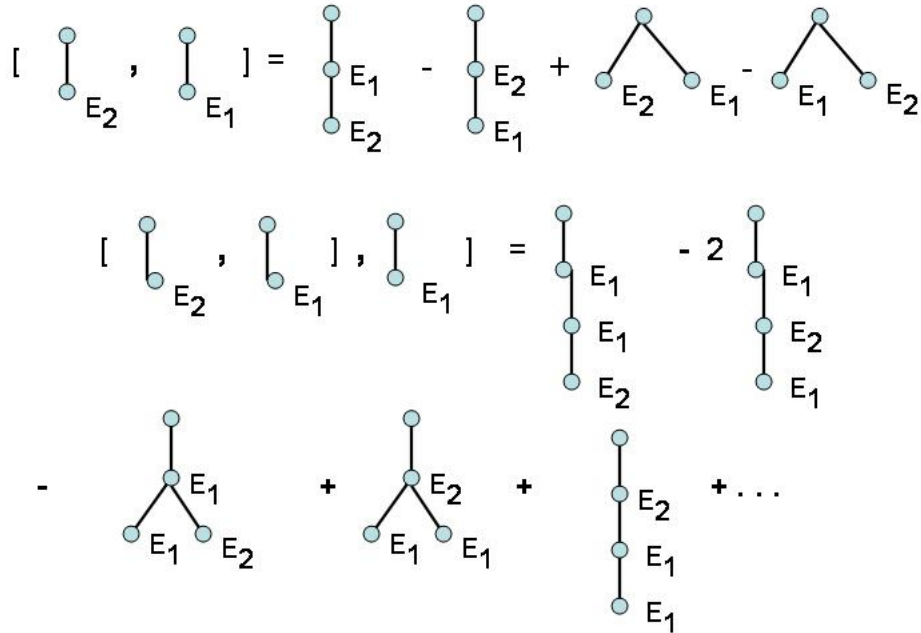


Figure 3: This figure illustrates some of the trees which arise in Example 3.3. The Lie bracket $[[E_2, E_1], E_1]$ only shows some of the trees which arise in the expansion.

This definition is illustrated in Figure 4. Note that $u(E; T_1, \dots, T_k) = u(E; t(T_1, \dots, T_k))$. We will need the following lemma.

Lemma 3.5 *Let $E, F \in \mathcal{D}$. Then*

$$t(v(E), v(F)) = v(E) \cdot v(F) - u(F; v(E)).$$

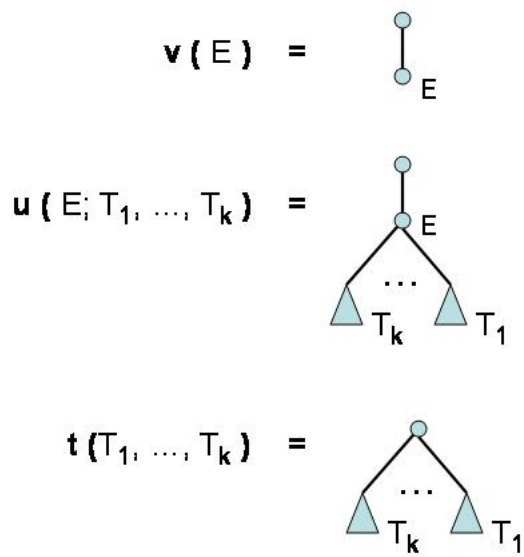


Figure 4: This figure illustrates the maps $v(E)$, $u(E; T_1, \dots, T_k)$ and $t(T_1, \dots, T_k)$ which are used to define an action of the algebra of labeled trees $k\{\mathcal{T}(\mathcal{D})\}$ on R .

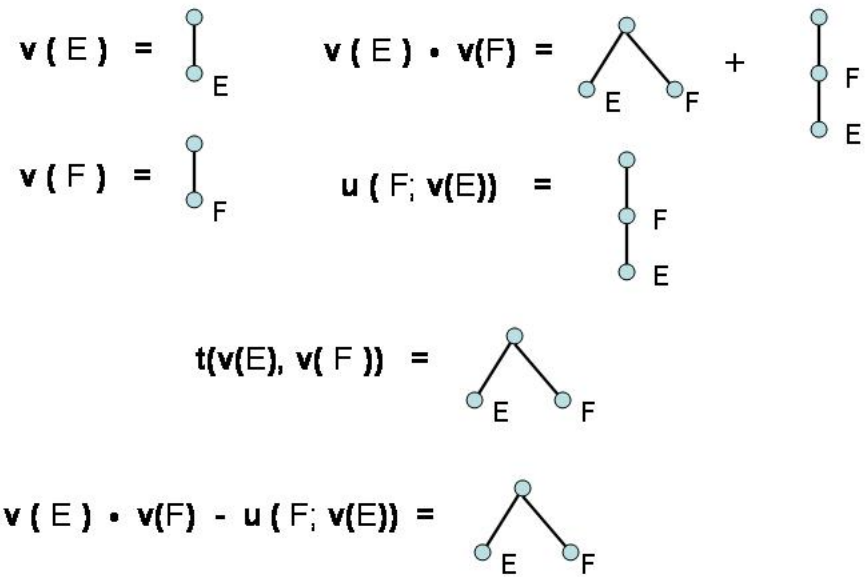


Figure 5: This figure illustrates the computation in Lemma 3.5.

PROOF: The proof of the lemma follows immediately from the definition of multiplication for trees.

This Lemma is illustrated in Figure 5.

Definition 3.6 *Let R be a commutative k -algebra, and let \mathcal{D} be a Lie algebra of derivations of R . A map $\mathcal{D} \times \mathcal{D} \rightarrow \mathcal{D}$ sending $(E, F) \in \mathcal{D} \times \mathcal{D}$ to $\nabla_E F \in \mathcal{D}$ satisfying*

- a) $\nabla_{E_1+E_2} F = \nabla_{E_1} F + \nabla_{E_2} F$
- b) $\nabla_E (F_1 + F_2) = \nabla_E F_1 + \nabla_E F_2$
- c) $\nabla_{f \cdot E} F = f \cdot \nabla_E F$
- d) $\nabla_E (f \cdot F) = f \cdot \nabla_E F + E(f)F$

where $E, F \in \mathcal{D}$, $f \in R$, is called a connection.

(For example, M could be a C^∞ -manifold, R be the algebra of C^∞ functions on M , \mathcal{D} could be the Lie algebra of vector fields on M , and $\nabla_E F$ could be the Koszul connection (see [12], Chapter 5, and [13], Chapters 5 and 6).)

Construction 3.7 We use the action of \mathcal{D} on R and a connection $\mathcal{D} \times \mathcal{D} \rightarrow \mathcal{D}$ to construct an action of the algebra of labeled trees $k\{\mathcal{T}(\mathcal{D})\}$ on R . See Figures 6 and 7. We give an inductive description of the action. The description of this construction is intended to allow an inductive proof of Theorem 3.9. We make the following assumptions about the action.

- a) The tree $v(E)$ acts as E .
- b) The tree $u(E; v(F))$ acts as $\nabla_F E$.
- c) Suppose that T is a labeled ordered tree whose root has a single child and which acts on R as the differential operator E_T . Suppose further that U is a labeled ordered tree which contains T as a proper subtree. Denote by $U(T|v(E_T))$ the labeled ordered tree resulting from replacing the subtree T with the tree $v(E_T)$. In this construction, we require that U acts like $U(T|v(E_T))$.

This assumption says that a subtree whose root has one child can be replaced by a tree which has one non-root node which is labeled with a derivation whose action is that of the original subtree.

We make use of these assumptions as follows.

d) If $E, F \in \mathcal{D}$, by Lemma 3.5 the tree $t(v(E), v(F))$ acts as $v(E) \cdot v(F) - u(F; v(E))$, whose action we know by (a) and (b).

e) We define the action of a tree whose root has only one child by induction on the number of children of the child of the root:

- If the child of the root has one child, by induction on the number of nodes we know how that child acts, since application of (c) and (b) allows us to determine the action of the tree on R .
- Suppose that $T = u(E; T_1, \dots, T_{n+1})$, where each T_i is a tree whose root has only one child. Then

$$T_1 \cdot u(E; T_2, \dots, T_{n+1}) = t(T_1, u(E; T_2, \dots, T_{n+1})) + T + \sum_j U_j,$$

where the U_j are trees in which T_1 has been linked to various nodes (other than the root) in the trees T_j in $t(E; T_2, \dots, T_{n+1})$. By induction we know the action of T_1 , of $u(E; T_2, \dots, T_{n+1})$, and of the U_j on R . By (c) and (d) we know the action of $t(T_1, u(E; T_2, \dots, T_{n+1}))$. Therefore we know the action of T .

This gives the action of a tree whose root has only one child.

f) We now determine the action of a general tree by induction on the number of children of the root:

- The case where the root has one child follows from (e).
- Suppose that $T = t(T_1, \dots, T_{n+1})$, where each T_i is a tree whose root has one child. Now

$$T_1 \cdot t(T_2, \dots, T_{n+1}) = T + \sum_j V_j,$$

where the V_j are trees whose roots have n children. We know the action of T_1 by (e), and of $t(T_2, \dots, T_{n+1})$ and V_j by induction. Therefore we know the action of T .

Note that this construction includes Example 3.2, upon letting

$$\nabla_{E_i} E_j = \sum_{\mu_i, \mu_j} r_{i\mu_i} \frac{\partial r_{j\mu_j}}{\partial X_{\mu_i}} \frac{\partial}{\partial X_{\mu_j}}$$

where the E_i are the differential operators given in (1). This follows from Definition 3.6 and the fact that

$$\nabla_{\frac{\partial}{\partial X_i}} \frac{\partial}{\partial X_j} = 0.$$

Note that Construction 3.7 gives the basic description of the action of a bialgebra of ordered trees whose non-root nodes are labeled with derivations of R on the commutative algebra R .

Definition 3.8 *If $T_1, T_2 \in k\{\mathcal{T}(\mathcal{D})\}$ act identically on R for $T_1, T_2 \in k\{\mathcal{T}(\mathcal{D})\}$, write $T_1 \sim T_2$.*

Theorem 3.9 *Let R be a commutative k -algebra, and let $\nabla_E F$ be a connection on the Lie algebra \mathcal{D} of derivations of R . Then Construction 3.7 gives a $k\{\mathcal{T}(\mathcal{D})\}$ -module structure on R . This module structure induces a map $\psi : k\{\mathcal{T}(\mathcal{D})\} \rightarrow \text{End}(R)$. The following conditions are satisfied:*

1. $h \cdot r = \psi(h) \cdot r$ for all $h \in k\{\mathcal{T}(\mathcal{D})\}$, $r \in R$;
2. the $k\{\mathcal{T}(\mathcal{D})\}$ -module structure on R is a $k\{\mathcal{T}(\mathcal{D})\}$ -module algebra structure;
3. $\text{Im } \psi \subseteq \text{Diff}(R)$.

PROOF:

1. Assumption (a) in Construction 3.7 guarantees that the actions of h and $\psi(h)$ are the same if h is a tree with one non-root node.

Assumption (b) describes how a tree with two nodes whose root has one child acts.

Condition (c) says that subtrees which act as derivations can be replaced by trees with one non-root node which act as the same derivation.

Condition (d) says that the actions of h and $\psi(h)$ are the same if h is the product of two trees which have only one non-root node.

Conditions (e) and (f) are used to prove the result by induction on the number of children of the root of any tree, and on the number of children of the child of the root in the case that the root has only one child.

2. The fact that the action gives a $k\{\mathcal{T}(\mathcal{D})\}$ -module algebra structure follows from the definition of the coalgebra structure of $k\{\mathcal{T}(\mathcal{D})\}$.
3. $\text{Diff}(R)$ is generated by derivations. Since Construction 3.7 gives the action of any tree as a sum of products of derivations, it follows that $\psi(h) \in \text{Diff}(R)$.

$$\begin{aligned}
 \mathbf{v}(E)(r) &= \begin{array}{c} \circ \\ | \\ \circ \\ \mathbf{E} \end{array} (r) \quad \mapsto \quad \mathbf{E}(r) \\
 \\
 \mathbf{u}(E; \mathbf{v}(F))(r) &= \begin{array}{c} \circ \\ | \\ \circ \\ \mathbf{E}(r) \\ | \\ \circ \\ \mathbf{F} \end{array} \quad \mapsto \quad \nabla_{\mathbf{F}} \mathbf{E}(r)
 \end{aligned}$$

Figure 6: An illustration of the action defined by Construction 3.7.

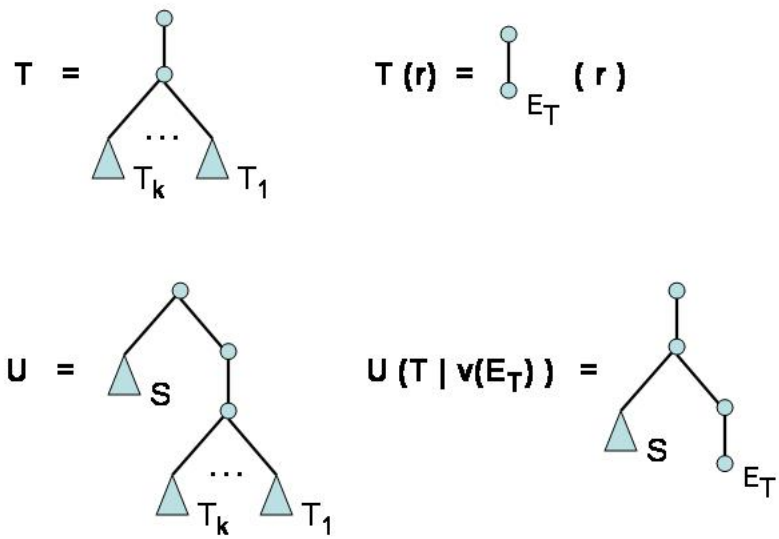


Figure 7: Another illustration of the action defined by Construction 3.7.

Definition 3.10 Let T be a labeled ordered tree. Suppose that a non-root node i of T is labeled with rE , where $r \in R$ and $E \in \mathcal{D}$. Denote the labeled ordered subtree whose (unlabeled) root is i by T_i . Denote by $T(i, G, T')$ the tree identical to T , except that the node i of $T(i, G, T')$ is labeled with $G \in \mathcal{D}$, and the labeled ordered subtree rooted at node i is $T' \in k\{\mathcal{T}(\mathcal{D})\}$. (Note that $T(i, rE, T_i) = T$.) Extending this notation allows us to replace T_i with a linear combination of trees.

Let R be a commutative k -algebra, let $\mathcal{D} = \text{Der}(R)$, and suppose that R is a Hopf module algebra over the k -bialgebra $k\{\mathcal{T}(\mathcal{D})\}$. The $k\{\mathcal{T}(\mathcal{D})\}$ -module algebra structure of R is called Leibnitz if

$$T \cdot s = \sum_{(T_i)} (T_{i(1)} \cdot r)(T(i, E, T_{i(2)}) \cdot s)$$

for all trees T , for all non-root nodes i of T , and for all factorizations rE , where $r \in R$, $E \in \mathcal{D}$, of the label of node i . (Note the coproduct in the formula above is over the tree T_i , not over the tree T .)

Basically the $k\{\mathcal{T}(\mathcal{D})\}$ -module algebra structure is Leibnitz if the action of a labeled tree is consistent in that subtrees act consistently with how they act as separate trees. See Figure 8 for a simple example.

Theorem 3.11 Let R be a commutative k -algebra, and $\mathcal{D} = \text{Der}(R)$, and let $\nabla : \mathcal{D} \times \mathcal{D} \rightarrow \mathcal{D}$ be a connection. Then the $k\{\mathcal{T}(\mathcal{D})\}$ -module algebra structure on R given in Construction 3.7 is Leibnitz.

PROOF: Let $T \in k\{\mathcal{T}(\mathcal{D})\}$, and suppose that the subtree rooted at node i is $T_i = t(T_{i1}, \dots, T_{ik})$, where each T_{ij} is a tree whose root has only one child. By assumption (c) of Construction 3.7 the subtree T_{ij} can be replaced by $U_j = v(F_j)$ where T_{ij} acts as F_j , so that T_i can be replaced by $U = t(U_1, \dots, U_k)$.

We show by induction on k that

$$T \cdot s = \sum_{(U)} (U_{(1)} \cdot r)(T(i, E, U_{(2)}) \cdot s).$$

If $k = 1$ then let V be the subtree rooted at parent node of i which has node i as its one child, which is labeled with rE and which has one child node labeled with F_1 , that is, the tree $u(rE; v(F_1))$. By assumption (b) of Construction 3.7, the subtree V acts as

$$\nabla_{F_1}(rE) = r\nabla_{F_1}E + F_1(r)E$$

so that

$$T \cdot s = \sum_{(U)} (U_{(1)} \cdot r) (T(i, E, U_{(2)}) \cdot s)$$

in this case.

Now consider the case $k > 1$. Define $G_\ell = \nabla_{F_1} F_\ell$. Here

$$\begin{aligned}
& \mathbf{u}(rE; \mathbf{v}(F_1), \dots, \mathbf{v}(F_k)) \\
& \sim \mathbf{v}(F_1) \cdot \mathbf{u}(rE; \mathbf{v}(F_2), \dots, \mathbf{v}(F_k)) \\
& \quad - \mathbf{t}(\mathbf{v}(F_1), \mathbf{u}(rE; \mathbf{v}(F_2), \dots, \mathbf{v}(F_k))) \\
& \quad - \sum_{\ell=2}^k \mathbf{u}(rE; \mathbf{v}(F_2), \dots, \mathbf{v}(G_\ell), \mathbf{v}(F_k)) \\
& \sim \mathbf{v}(F_1) \cdot \left(\sum_{(V)} (V_{(1)} \cdot r) (\mathbf{u}(E; V_{(2)})) \right) \tag{2} \\
& \quad - \sum_{(V)} (V_{(1)} \cdot r) (\mathbf{t}(\mathbf{v}(F_1), \mathbf{u}(E; V_{(2)}))) \\
& \quad - \sum_{\ell, (V)} (V_{\ell(1)} \cdot r) (\mathbf{u}(E; V_{\ell(2)})) \\
& \sim \sum_{(U)} (U_{(1)} \cdot r) (\mathbf{u}(E; U_{(2)})) \\
& \quad + \sum_{(V)} (V_{(1)} \cdot r) (\mathbf{t}(\mathbf{v}(F_1), \mathbf{u}(E; V_{(2)}))) \\
& \quad + \sum_{\ell, (V)} (V_{\ell(1)} \cdot r) (\mathbf{u}(E; V_{\ell(2)})) \\
& \quad - \sum_{(V)} (V_{(1)} \cdot r) (\mathbf{t}(\mathbf{v}(F_1), \mathbf{u}(E; V_{(2)}))) \\
& \quad - \sum_{\ell, (V)} (V_{\ell(1)} \cdot r) (\mathbf{u}(E; V_{\ell(2)})) \\
& \sim \sum_{(U)} (U_{(1)} \cdot r) (\mathbf{u}(E; U_{(2)}))
\end{aligned}$$

(Note that we are using V as a “local variable” in each sum, and that the value varies, depending on context.) The identity of the actions of the terms in expression (2) follows from the induction hypothesis. Since $\mathbf{u}(rE; \mathbf{v}(F_1), \dots, \mathbf{v}(F_k)) \sim \sum_{(U)} (U_{(1)} \cdot r) (\mathbf{u}(E; U_{(2)}))$ the theorem now follows from assumption (c) of Construction 3.7.

$$\begin{aligned}
\mathbf{T} \bullet \mathbf{s} &= \left(\begin{array}{c} \circ \\ | \\ \circ \text{ } rE \\ | \\ \circ \text{ } F \end{array} \right) \bullet \mathbf{s} \\
\mathbf{T}_i &= \begin{array}{c} \circ \\ | \\ \circ \text{ } F \end{array} \quad \Delta \left(\begin{array}{c} \circ \\ | \\ \circ \text{ } F \end{array} \right) = 1 \otimes \begin{array}{c} \circ \\ | \\ \circ \text{ } F \end{array} + \begin{array}{c} \circ \\ | \\ \circ \text{ } F \end{array} \otimes 1 \\
\mathbf{T}(i, E, \begin{array}{c} \circ \\ | \\ \circ \text{ } F \end{array}) &= \begin{array}{c} \circ \\ | \\ \circ \text{ } E \\ | \\ \circ \text{ } F \end{array} \quad \mathbf{T}(i, E, 1) = \begin{array}{c} \circ \\ | \\ \circ \text{ } E \end{array} \\
\mathbf{T} \bullet \mathbf{s} &= (1 \bullet r) \left(\begin{array}{c} \circ \\ | \\ \circ \text{ } E \\ | \\ \circ \text{ } F \end{array} \mathbf{s} \right) + \left(\begin{array}{c} \circ \\ | \\ \circ \text{ } F \\ | \\ \circ \end{array} r \right) \left(\begin{array}{c} \circ \\ | \\ \circ \text{ } E \end{array} \mathbf{s} \right)
\end{aligned}$$

Figure 8: This figure illustrates the defining formula for a Leibnitz H -module algebra structure for a simple tree acting on a function s . Note that in this simple case, the formula defining a Leibnitz structure generalizes the following standard formula for connections: $(\nabla_F(rE))(s) = r((\nabla_F E)s) + F(r)E(s)$.

4 R/k -bialgebras

The notions of R/k -bialgebra and R/k -Hopf algebra as described in [9] and [10] capture many of the essential aspects of differential algebra. We first review some of the material found there.

Let R be a k -algebra. A R/k -algebra is a k -algebra B into which R is embedded. Note that this makes B into a left and right R -module, and that $(rb)s = r(bs)$ for all $r, s \in R, b \in B$, and that also $(rb)c = r(bc)$, $(br)c = b(rc)$, and $(bc)r = b(cr)$ for all $b, c \in B, r \in R$. When we refer to the R -module structure of B , we will understand the left R -module structure.

We denote by $B \otimes_R B$ the tensor product of B with itself using the left R -module structure of B . That is, with $(rb) \otimes c = b \otimes (rc)$. Note that in general $B \otimes_R B$ is not an R -algebra.

The material we present here is related to the \times_R -bialgebra construction given in [15]. There a \times_R -bialgebra is defined in terms of maps between B and $B \times_R B$, but the space $B \times_R B$ in [15] is only a k -subspace of $B \otimes_R B$.

Definition 4.1 *A R/k -bialgebra is a R/k -algebra B together with R -module maps $\Delta : B \rightarrow B \otimes_R B$ and $\epsilon : B \rightarrow R$ satisfying*

- a) B together with the maps Δ and ϵ is a coalgebra over R .
- b) $\Delta(1) = 1 \otimes 1$.
- c) For all $b, c \in B$, if $\Delta(b) = \sum_i b_i \otimes b'_i$ and $\Delta(c) = \sum_j c_j \otimes c'_j$ are any representations of $\Delta(b), \Delta(c) \in B \otimes_R B$, then $\Delta(bc) = \sum_{i,j} b_i c_j \otimes b'_i c'_j$.
- d) $\epsilon(1) = 1$.
- e) $\epsilon(bc) = \epsilon(b\epsilon(c))$.

Note that condition (c) of Definition 4.1 implies that

$$\Delta(br) = \sum_{(b)} b_{(1)}r \otimes b_{(2)} = \sum_{(b)} b_{(1)} \otimes b_{(2)}r$$

for $b \in B$ and $r \in R$. It can be shown (see [9] for details) that conditions (b) and (c) are equivalent to the assertion that the action of B on $B \otimes_R B$ defined by $b \cdot (c \otimes d) \stackrel{\text{def}}{=} \Delta(b)(c \otimes d)$, for $b, c, d \in B$, gives $B \otimes_R B$ a well-defined left

B -module structure, and that conditions (d) and (e) are equivalent to the assertion that the action of B defined on R by $b \cdot r \stackrel{\text{def}}{=} \epsilon(br)$, for $b \in B$ and $r \in R$, gives R a well-defined left B -module structure.

If B is a R/k -algebra, then B is a (R, R) -bimodule via the left and right actions of R on B induced by the embedding of R in B . Denote by $B \otimes_r B$ the tensor product of B with itself using this (R, R) -bimodule structure. That is, $B \otimes_r B$ is an (R, R) -bimodule with $r(b \otimes_r c) = (rb) \otimes_r c$, $(b \otimes_r c)r = b \otimes_r (cr)$, and that $br \otimes_r c = b \otimes_r rc$. The multiplication on B induces a map $\mu : B \otimes_r B \rightarrow B$.

Definition 4.2 *Let B be a R/k -bialgebra. An antiproduct for B is a k -linear map $E : B \rightarrow B \otimes_r B$ satisfying*

- a) $E(rb) = rE(b) = E(b)r$ for all $r \in R, b \in B$;
- b) $\sum_{(b)} E(b_{(1)})b_{(2)} = b \otimes_r 1$ for all $b \in B$;
- c) $(I \otimes_R E) \circ \Delta(b) = (\Delta \otimes_r I) \circ E(b)$ for all $b \in B$;
- d) $\mu \circ E(b) = \epsilon(b)1$ for all $b \in B$.

A R/k -bialgebra which has an antiproduct is called a R/k -Hopf algebra.

We recall from [9, Proposition 8]

Proposition 4.3 *Let R be a k -algebra. If H is a cocommutative k -Hopf algebra over which R is a H -bimodule algebra, then $R \#_k H \stackrel{\text{def}}{=} R \otimes_k H$ is a R/k -Hopf algebra, with the R -coalgebra structure given by*

$$R \otimes_k H \xrightarrow{I_H \otimes \Delta_H} R \otimes_k H \otimes_k H \cong (R \otimes_k H) \otimes_R (R \otimes_k H)$$

and with multiplication given by

$$(r \# h)(s \# k) \stackrel{\text{def}}{=} \sum_{(h)} r(h_{(1)} \cdot s) \# h_{(2)}k$$

where $r, s \in R, h, k \in H$, and with antiproduct given by

$$E(r \# h) = \sum_{(h)} (r \# h_{(1)}) \otimes_r (1 \# S(h_{(2)}))$$

where $r \in R, h \in H$, and S is the antipode of H .

Proposition 4.4 *Let R be a k -algebra, let H and \bar{H} be cocommutative k -Hopf algebras over which R is a module algebra, and let $\varphi : H \rightarrow \bar{H}$ be a k -Hopf algebra homomorphism which is consistent with the H and \bar{H} -module algebra structures on R . Then the map*

$$R \#_k H \rightarrow R \#_k \bar{H}$$

given by $r \# h \mapsto r \# \varphi(h)$ is a R/k -Hopf algebra homomorphism.

PROOF: Omitted.

Theorem 4.5 *Let R be a commutative k -algebra, let $\mathcal{D} = \text{Der}(R)$, and let $\mathcal{S} \subseteq \mathcal{D}$.*

- a) *Let $\text{Diff}_k(\mathcal{S})$ denote the k -algebra of higher order derivations generated by \mathcal{S} , that is, the subalgebra of $U(\mathcal{D})$ generated by \mathcal{S} . Then the smash product $R \#_k \text{Diff}_k(\mathcal{S})$ is a R/k -Hopf algebra. The subalgebra of $\text{Diff}(R)$ generated by \mathcal{S} is a homomorphic image of $R \#_k \text{Diff}_k(\mathcal{S})$.*
- b) *Let $k\{\mathcal{T}(\mathcal{S})\}$ denote the Hopf algebra of trees labeled with elements of \mathcal{S} defined in Section 2. Then the smash product $R \#_k k\{\mathcal{T}(\mathcal{S})\}$ is a R/k -Hopf algebra.*
- c) *The map*

$$R \#_k k\{\mathcal{T}(\mathcal{S})\} \rightarrow R \#_k \text{Diff}_k(\mathcal{S})$$

is a R/k -bialgebra homomorphism.

PROOF: The action of $k\{\mathcal{T}(\mathcal{D})\}$ on R given in Construction 3.7 defines a Hopf algebra homomorphism $k\{\mathcal{T}(\mathcal{D})\} \rightarrow \text{Diff}(R)$ which is consistent with the Hopf module-algebra structures on R , and so induces a R/k -Hopf algebra homomorphism $R \#_k k\{\mathcal{T}(\mathcal{D})\} \rightarrow R \#_k \text{Diff}(R)$, which allows us to use the differential algebra structure on $R \#_k k\{\mathcal{T}(\mathcal{D})\}$ to study the differential algebra structure on $R \#_k \text{Diff}(R)$

$\text{Diff}_k(\mathcal{S})$ is a cocommutative k -Hopf algebra, so $R \#_k \text{Diff}_k(\mathcal{S})$ is a R/k -Hopf algebra.

By Proposition 2.1, $k\{\mathcal{T}(\mathcal{S})\}$ is a cocommutative k -Hopf algebra. Therefore, by Proposition 4.3, $R \#_k k\{\mathcal{T}(\mathcal{S})\}$ is a R/k -Hopf algebra.

Part (c) follows immediately from Proposition 4.4.

If R is a k -bialgebra (for example if R is the coordinate ring of an affine algebraic group), then the set $\text{Diff}^R(R)$ of right invariant differential operators is a cocommutative k -Hopf algebra. The following proposition follows immediately from [7, Theorem 2.4.5].

Proposition 4.6 *Let R be a k -bialgebra, and let $\text{Diff}^R(R)$ be the k -Hopf algebra of right invariant differential operators. Then*

$$\text{Diff}(R) \cong R \#_k \text{Diff}^R(R)$$

is a R/k -Hopf algebra.

If R is a k -bialgebra, and if \mathcal{B} is a k -basis for $\text{Der}^R(R)$, the Lie algebra of right invariant derivations of R to itself, then we will see that Proposition 4.6 allows us to use the smash product $R \#_k k\{\mathcal{T}(\mathcal{B})\}$ to do formal computations in $\text{Diff}(R)$. (For example, such a basis always exists if R is the coordinate ring of an affine algebraic group.)

We can use the R/k -bialgebra $R \#_k k\{\mathcal{T}(\mathcal{S})\}$ to do formal computations involving elements of a subset $\mathcal{S} \subseteq \text{Der}(R)$.

Let $\mathcal{S} \subseteq \text{Der}(R)$, and let $F(\mathcal{S})$ be the free associative algebra generated by \mathcal{S} . Recalling that the elements of \mathcal{S} are primitive, we get a Hopf algebra structure on $F(\mathcal{S})$. Since $F(\mathcal{S})$ is freely generated by \mathcal{S} , we have maps $F(\mathcal{S}) \rightarrow \text{Diff}_k(\mathcal{S})$ mapping $E \in \mathcal{S}$ to $E \in \text{Diff}_k(\mathcal{S})$, and $F(\mathcal{S}) \rightarrow k\{\mathcal{T}(\mathcal{S})\}$ mapping $E \in \mathcal{S}$ to $v(E) \in k\{\mathcal{T}(\mathcal{S})\}$. These induce maps $p : R \#_k F(\mathcal{S}) \rightarrow R \#_k \text{Diff}_k(\mathcal{S})$ and $i : R \#_k F(\mathcal{S}) \rightarrow R \#_k k\{\mathcal{T}(\mathcal{S})\}$. If R is an algebra for which there is a connection on $\text{Der}(R)$ (for example, if R is the algebra of C^∞ functions on a Riemannian manifold), there is a map $\varphi : R \#_k k\{\mathcal{T}(\mathcal{S})\} \rightarrow R \#_k \text{Diff}_k(\mathcal{S})$ induced by the map described in Construction 3.7. We have

Theorem 4.7 *Let R be a commutative algebra for which there is a connection on $\text{Der}(R)$, and let p , i , and φ be the maps described above. Then the diagram*

$$\begin{array}{ccc}
 & R \#_k k\{\mathcal{T}(\mathcal{S})\} & \\
 & \nearrow i & \searrow \varphi \\
 R \#_k F(\mathcal{S}) & \xrightarrow{p} & R \#_k \text{Diff}_k(\mathcal{S})
 \end{array}$$

commutes.

This theorem allows us to do formal computations involving elements of \mathcal{S} in the algebra $R \#_k k\{\mathcal{T}(\mathcal{S})\}$ rather than in $R \#_k F(\mathcal{S})$.

5 Quotients of R/k -bialgebras

In this section we discuss certain quotients of the R/k -Hopf algebra $R \#_k k\{\mathcal{T}(\mathcal{D})\}$, where $\mathcal{D} = \text{Der}(R)$. The main result in this section is Theorem 5.8, which says that a Leibnitz action of the R/k -Hopf algebra $R \#_k k\{\mathcal{T}(\mathcal{D})\}$ can be computed from the action of $R \#_k k\{\mathcal{T}(\mathcal{B})\}$ if \mathcal{B} is an R -basis of \mathcal{D} .

Let B be a R/k -bialgebra. A R/k -biideal is an ideal I in the R/k -algebra B , such that $\epsilon(I) = 0$, and such that if $\pi : B \rightarrow B/I$ is the projection of B onto B/I , we have $(\pi \otimes_R \pi) \circ \Delta(I) = 0$. If I is a R/k -biideal in B , then B/I is a R/k -bialgebra. If B is a R/k -Hopf algebra with antiproduct E , and if $(\pi \otimes_r \pi) \circ E(I) = 0$, then I is called a R/k -Hopf ideal, and B/I is a R/k -Hopf algebra.

We will use the following definition in the sequel.

Definition 5.1 *Let R be a Leibnitz $k\{\mathcal{T}(\mathcal{D})\}$ -module algebra. Let $\mathcal{I}(k\{\mathcal{T}(\mathcal{D})\})$ be the R -linear span of the elements of the form*

$$1 \# T - \sum_{(T_i)} T_{i(1)} \cdot r \# T(i, E, T_{i(2)}), \quad (3)$$

where $T \in k\{\mathcal{T}(\mathcal{D})\}$, with non-root node i labeled with rE , with $r \in R$, $E \in \mathcal{D}$ (we include all possible factorizations of the label of node i).

Lemma 5.2 *The subspace $\mathcal{I}(k\{\mathcal{T}(\mathcal{D})\})$ defined in Definition 5.1 is a two-sided ideal in $R \#_k k\{\mathcal{T}(\mathcal{D})\}$.*

PROOF: Let J be the ideal of $R \#_k k\{\mathcal{T}(\mathcal{D})\}$ generated by $\mathcal{I}(k\{\mathcal{T}(\mathcal{D})\})$.

Let Z be an element of the form (3), and let T' be any tree. It follows from the definition of the product of trees that $Z(1 \# T')$ is an R -linear combination of elements of the form (3).

We now show that

$$\begin{aligned} (1 \# T')Z &= 1 \# T' \cdot (1 \# T - \sum_{(T_i)} T_{i(1)} \cdot r \# T(i, E, T_{i(2)})) \\ &= 1 \# T' \cdot T - \sum_{(T'), (T_i)} T'_{(1)} \cdot T_{i(1)} \cdot r \# T'_{(2)} \cdot T(i, E, T_{i(2)}) \end{aligned}$$

is an R -linear combination of elements of the form (3). Since $k\{\mathcal{T}(\mathcal{D})\}$ is generated as an algebra by trees whose root has one child (see [4]), it is sufficient to show this in the case that the root of the tree T' has one child. Terms in the tree product $T' \cdot T$ pair in an obvious fashion with terms in $\sum_{(T'),(T_i)} T'_{(1)} \cdot T_{i(1)} \cdot r \# T'_{(2)} \cdot T(i, E, T_{i(2)})$. Therefore $(1 \# T') \cdot Z$ is again an R -linear combination of elements of the form (3). This completes the proof of the lemma.

Proposition 5.3 *Let R be a commutative k -algebra, let $\mathcal{D} = \text{Der}(R)$, and suppose that R is a Leibnitz $k\{\mathcal{T}(\mathcal{D})\}$ -module algebra. Then the ideal $\mathcal{I}(k\{\mathcal{T}(\mathcal{D})\})$ is a R/k -Hopf ideal.*

PROOF:

It is immediate that ϵ is zero on any element of the form (3), since it is zero on any tree with more than one node.

Let π be the projection of $k\{\mathcal{T}(\mathcal{D})\}$ onto $k\{\mathcal{T}(\mathcal{D})\}/\mathcal{I}(k\{\mathcal{T}(\mathcal{D})\})$. To see that $(\pi \otimes_R \pi) \circ \Delta$ is zero on any element of the form (3), write $\Delta(T) = \sum_j T'_j \otimes T''_j \in k\{\mathcal{T}(\mathcal{D})\} \otimes_k k\{\mathcal{T}(\mathcal{D})\}$. If node i occurs in T'_j , then the corresponding term arising in the coproduct applied to element (3) is

$$\begin{aligned} & 1 \# T'_j \otimes_R 1 \# T''_j - \sum_{(T_i)} T_{i(1)} \cdot r \# T'_j(i, E, T_{i(2)}) \otimes_R 1 \# T''_j \\ &= (1 \# T'_j - \sum_{(T_i)} T_{i(1)} \cdot r \# T'_j(i, E, T_{i(2)})) \otimes_R 1 \# T''_j, \end{aligned}$$

which is clearly in $\text{Ker}(\pi \otimes_R I) \subseteq \text{Ker}(\pi \otimes_R \pi)$. Similarly, if node i occurs in tree T''_j , then the corresponding term of the coproduct applied to this element is in $\text{Ker}(I \otimes_R \pi) \subseteq \text{Ker}(\pi \otimes_R \pi)$. It follows that $(\pi \otimes_R \pi) \circ \Delta$ vanishes on I , since it vanishes on a generating set for it as an ideal in the algebra $R \#_k k\{\mathcal{T}(\mathcal{D})\}$.

To show that $\mathcal{I}(k\{\mathcal{T}(\mathcal{D})\})$ is compatible with the antiproduct of $k\{\mathcal{T}(\mathcal{D})\}$, we will need to work with a restricted generating set of $\mathcal{I}(k\{\mathcal{T}(\mathcal{D})\})$.

Lemma 5.4 *Let R be a commutative k -algebra, and let $\mathcal{D} = \text{Der}(R)$. Suppose that we are given a Leibnitz $k\{\mathcal{T}(\mathcal{D})\}$ -module action on R . Let J be the ideal in $R \#_k k\{\mathcal{T}(\mathcal{D})\}$ generated by all elements of the form (3) where T ranges over all labeled ordered trees whose root has only one child. Then $J = \mathcal{I}(k\{\mathcal{T}(\mathcal{D})\})$.*

PROOF: It is immediate that $J \subseteq \mathcal{I}(k\{\mathcal{T}(\mathcal{D})\})$. Let T be any labeled ordered tree, and let i be a node of T which is labeled with rE . We will prove that

$$1 \# T - \sum_{(T_i)} T_{i(1)} \cdot r \# T(i, E, T_{i(2)}) \quad (4)$$

is in J by induction on the number of children of the root of T .

If the root of T has one child then the element (4) is in J by definition.

Suppose that the root of T has $n+1$ children. Let T_0 be the tree consisting of the tree whose root has one child, which is the first child of the root of the tree T , in the order in which they occur in T . Let T_1 be the tree whose root has as children all of the other children of the root of the tree T and their descendants, in the order in which they occur in T . The root of the tree T_1 has n children. To show that element (4) is in J , we consider two cases.

Node $i \in T_0$

In this case let $T_0 \cdot T_1 = T + \sum_j U_j$, where each U_j is a tree whose root has only n children. Then, since $T = T_0 \cdot T_1 - \sum_j U_j$,

$$\begin{aligned} 1 \# T - \sum_{(T_i)} T_{i(1)} \cdot r \# T(i, E, T_{i(2)}) \\ &= (1 \# T_0 - \sum_{(T_i)} T_{i(1)} \cdot r \# T_0(i, E, T_{i(2)})) \cdot (1 \# T_1) \\ &\quad - \sum_j (1 \# U_j - \sum_{(T_i)} T_{i(1)} \cdot r \# U_j(i, E, T_{i(2)})). \end{aligned}$$

Note that, since the tree T_0 precedes T_1 in the product, the labeled ordered subtree T_i rooted at node i is the same in T_0 as in U_j . The term

$$(1 \# T_0 - \sum_{(T_i)} T_{i(1)} \cdot r \# T_0(i, E, T_{i(2)})) \cdot (1 \# T_1)$$

is in J since the root of T_0 has one child and J is an ideal. The terms

$$1 \# U_j - \sum_{(T_i)} T_{i(1)} \cdot r \# U_j(i, E, T_{i(2)})$$

are in J by induction on the number of children of the root of the tree.

Node $i \in T_1$

In this case first note that

$$1 \# T_1 - \sum_{(V)} V_{(1)} \cdot r \# T_1(i, E, V_{(2)}), \quad (5)$$

where V is the labeled subtree of T_1 rooted at node i , is in J by induction, since the root of T_1 has only n children. Let

$$T_0 \cdot T_1 = T + \sum_k U_k + \sum_\ell U'_\ell,$$

where the U_k are the trees in the product in which T_0 is not attached to the node i or to any of its descendents, and the U'_ℓ are the trees in the product in which T_0 is attached to the node i or to one of its descendents. Now the element

$$\begin{aligned} & (1 \# T_0) \cdot (1 \# T_1 - \sum_{(V)} V_{(1)} \cdot r \# T_1(i, V_{(2)}, E)) \\ &= 1 \# T_0 \cdot T_1 - \sum_{(V)} T_0 \cdot V_{(1)} \cdot r \# T_1(i, V_{(2)}, E) \\ & \quad - \sum_{(V)} V_{(1)} \cdot r \# T_0 \cdot T_1(i, V_{(2)}, E) \\ &= 1 \# T + \sum_k 1 \# U_k + \sum_\ell 1 \# U'_\ell \\ & \quad - \sum_{(V)} T_0 \cdot V_{(1)} \cdot r \# T_1(i, V_{(2)}, E) \\ & \quad - \sum_{(V)} V_{(1)} \cdot r \# T(i, V_{(2)}, E) \\ & \quad - \sum_{(V)} V_{(1)} \cdot r \# U_k(i, V_{(2)}, E) \\ & \quad - \sum_{(V)} V_{(1)} \cdot r \# T_1(i, T_0 \cdot V_{(2)}, E) \\ &= 1 \# T - \sum_{(V)} V_{(1)} \cdot r \# T(i, V_{(2)}, E) \end{aligned} \quad (6)$$

$$+ \sum_k \left(1 \# U_k - \sum_{(V)} V_{(1)} \cdot r \# U_k(i, V_{(2)}, E) \right) \quad (7)$$

$$+ \sum_\ell \left(1 \# U'_\ell - \sum_{(V')} V'_{(1)} \cdot r \# U'_\ell(i, V'_{(2)}, E) \right) \quad (8)$$

where V' is the labeled subtree rooted at node i in U'_ℓ , is in the ideal J , since the element (5) is in the ideal. Since the roots of the trees U_k

and U'_ℓ have only n children, by induction the terms (7) and (8) are in J . Therefore the term (6) is in the ideal J .

This completes the proof of the lemma.

We now use Lemma 5.4 to show that $\mathcal{I}(k\{\mathcal{T}(\mathcal{D})\})$ is compatible with the antiproduct. Denote by π the projection of $R \#_k k\{\mathcal{T}(\mathcal{D})\}$ onto $R \#_k k\{\mathcal{T}(\mathcal{D})\}/\mathcal{I}(k\{\mathcal{T}(\mathcal{D})\})$. We need to show that $(\pi \otimes_r \pi) \circ E(\mathcal{I}(k\{\mathcal{T}(\mathcal{D})\})) = 0$. By [9, Proposition 6], since $R \#_k k\{\mathcal{T}(\mathcal{D})\}$ is pointed, it is sufficient to show that $(\pi \otimes_r \pi) \circ E = 0$ on a generating set of the ideal $\mathcal{I}(k\{\mathcal{T}(\mathcal{D})\})$. By Lemma 5.4 it is sufficient to consider the value of $(\pi \otimes_r \pi) \circ E$ on elements of the form (3), when the root of the tree T has only one child. In this case

$$\begin{aligned}
& E(1 \# T - \sum_{(T_i)} T_{i(1)} \cdot r \# T(i, E, T_{i(2)})) \\
&= 1 \# T \otimes_r 1 \# 1 - 1 \# 1 \otimes_r 1 \# T \\
&\quad - \sum_{(T_i)} T_{i(1)} \cdot r \# T(i, E, T_{i(2)}) \otimes_r 1 \# 1 \\
&\quad + \sum_{(T_i)} T_{i(1)} \cdot r \# 1 \otimes_r 1 \# T(i, E, T_{i(2)}) \\
&= (1 \# T - \sum_{(T_i)} T_{i(1)} \cdot r \# T(i, E, T_{i(2)})) \otimes_r 1 \# 1 \\
&\quad - 1 \# 1 \otimes_r (1 \# T - \sum_{(T_i)} T_{i(1)} \cdot r \# T(i, E, T_{i(2)})),
\end{aligned}$$

and this is clearly annihilated by $\pi \otimes_r \pi$. This proves Proposition 5.3.

Definition 5.5 *Suppose that the $k\{\mathcal{T}(\mathcal{D})\}$ -module algebra structure of R is Leibnitz. Let \mathcal{B} be an R -basis for \mathcal{D} . Repeated applications of the substitution*

$$1 \# T = \sum_u \sum_{(T_i)} T_{i(1)} \cdot r_u \# T(i, X_u, T_{i(2)}), \quad (9)$$

where the node i in the tree T is labeled with $\sum_u r_u X_u$, with $r_u \in R$, $X_u \in \mathcal{B}$, gives a map $\alpha_{\mathcal{B}} : R \#_k k\{\mathcal{T}(\mathcal{D})\} \rightarrow R \#_k k\{\mathcal{T}(\mathcal{B})\}$.

Lemma 5.6 *Suppose that the $k\{\mathcal{T}(\mathcal{D})\}$ -module algebra structure of R is Leibnitz, and let \mathcal{B} be an R -basis for \mathcal{D} . Then the map $\alpha_{\mathcal{B}} : R \#_k k\{\mathcal{T}(\mathcal{D})\} \rightarrow R \#_k k\{\mathcal{T}(\mathcal{B})\}$ given in Definition 5.5 is well-defined.*

PROOF: It is sufficient to show that application of the substitution rule (9) to two nodes does not depend on the order in which the rule is applied to the nodes. If neither node is an ancestor of the other, then it follows immediately that the result is independent of the order in which the rule is applied. If one node is an ancestor of the other, then the independence of the result of the order of application follows immediately from the fact that the algebra module structure is Leibnitz. This completes the proof of the lemma.

Let $\beta_{\mathcal{B}} : R \#_k k\{\mathcal{T}(\mathcal{B})\} \rightarrow R \#_k k\{\mathcal{T}(\mathcal{D})\}$ be the inclusion map. Then $\alpha_{\mathcal{B}} \circ \beta_{\mathcal{B}}$ is the identity on $R \#_k k\{\mathcal{T}(\mathcal{B})\}$.

Lemma 5.7 *Suppose that the $k\{\mathcal{T}(\mathcal{D})\}$ -module algebra structure of R is Leibnitz, and let \mathcal{B} be an R -basis for \mathcal{D} . Then*

$$\text{Ker } \alpha_{\mathcal{B}} = \mathcal{I}(k\{\mathcal{T}(\mathcal{D})\}).$$

PROOF: Note that $\text{Ker } \alpha_{\mathcal{B}}$ is the linear span of elements of the form

$$1 \# T - \sum_{u, (T_i)} T_{i(1)} \cdot s_u \# T(i, X_u, T_{i(2)})$$

where the node i in the tree T is labeled with $\sum_u s_u X_u$, with $s_u \in R$, $X_u \in \mathcal{B}$. These elements are all of the form (3), which span $\mathcal{I}(k\{\mathcal{T}(\mathcal{D})\})$, so that $\text{Ker } \alpha_{\mathcal{B}} \subseteq \mathcal{I}(k\{\mathcal{T}(\mathcal{D})\})$.

The ideal $\mathcal{I}(k\{\mathcal{T}(\mathcal{D})\})$ is the linear span of elements of the form

$$\begin{aligned} & 1 \# T - \sum_{(T_i)} T_{i(1)} \cdot r \# T(i, E, T_{i(2)}) \\ &= 1 \# T - \sum_{u, (T_i)} (T_{i(1)} \cdot r)(T_{i(2)} \cdot s_u) \# T(i, X_u, T_{i(3)}) \\ &= 1 \# T - \sum_{u, (T_i)} T_{i(1)} \cdot (rs_u) \# T(i, X_u, T_{i(2)}), \end{aligned}$$

Where rE , r , $s_u \in R$, $E \in \mathcal{D}$, is a factorization of the label of a node i , $X_u \in \mathcal{B}$, and $E = \sum_u s_u X_u$, so that $\mathcal{I}(k\{\mathcal{T}(\mathcal{D})\}) \subseteq \text{Ker } \alpha_{\mathcal{B}}$.

This completes the proof of the lemma.

We have the following theorem, which says that we can use $R \#_k k\{\mathcal{T}(\mathcal{B})\}$ to do computations in $R \#_k k\{\mathcal{T}(\mathcal{D})\}$.

Theorem 5.8 *Let R be a commutative k -algebra. Assume that $\mathcal{D} = \text{Der}(R)$ is free as an R -module. Suppose that we have a Leibnitz $k\{\mathcal{T}(\mathcal{D})\}$ -module algebra structure on R . Let \mathcal{B} be an R -basis of \mathcal{D} . Then*

$$\alpha_{\mathcal{B}} : R \#_k k\{\mathcal{T}(\mathcal{D})\} / \mathcal{I}(k\{\mathcal{T}(\mathcal{D})\}) \cong R \#_k k\{\mathcal{T}(\mathcal{B})\}.$$

PROOF: From Lemma 5.7 $\text{Ker } \alpha_{\mathcal{B}} = \mathcal{I}(k\{\mathcal{T}(\mathcal{D})\})$ so that the map $\alpha_{\mathcal{B}}$ is injective.

From the fact that $\alpha_{\mathcal{B}} \circ \beta_{\mathcal{B}}$ is the identity on $R \#_k k\{\mathcal{T}(\mathcal{B})\}$ it follows the $\alpha_{\mathcal{B}}$ is surjective.

If \mathcal{B} is an R -basis of $\mathcal{D} = \text{Der}(R)$, then there is a bijection between the Leibnitz $k\{\mathcal{T}(\mathcal{D})\}$ -module algebra structures of R and the $k\{\mathcal{T}(\mathcal{B})\}$ -module algebra structures of R . According to [4, Theorem 5.1], $k\{\mathcal{T}(\mathcal{B})\}$ is freely generated as an associative algebra by the set \mathcal{X} of trees whose root has a single child, which are labeled with elements of \mathcal{B} . Therefore, there is a bijection between Leibnitz $k\{\mathcal{T}(\mathcal{D})\}$ -module algebra structures on R and functions from \mathcal{X} to $\text{Der}(R)$.

In particular, there are Leibnitz module algebra structures for the Hopf algebra of labeled ordered trees, labeled with elements of $\text{Der}(R)$, on $R = k[X_1, \dots, X_N]$ other than the example given in Example 3.2. In particular, there exist module algebra structures for the Hopf algebra of trees labeled with elements of $\{\partial/\partial X_1, \dots, \partial/\partial X_N\}$ under which trees with more than two nodes whose root has only one child act as non zero first-order differential operators. We will see in the next section that under an additional hypothesis, the $k\{\mathcal{T}(\mathcal{D})\}$ -module algebra structure is determined by the actions of trees with two nodes, which correspond to the actions of the elements of \mathcal{D} , and trees with three nodes whose roots have only a single child, which correspond to the connection on R .

6 Coherent actions and connections

In this section we discuss how certain actions of $k\{\mathcal{T}(\mathcal{D})\}$ on R are determined by the action of $E \in \mathcal{D} = \text{Der}(R)$, and of the action of the connection $\nabla_E F$ for $E, F \in \mathcal{D}$. Throughout this section R is a commutative k -algebra.

We consider actions of $k\{\mathcal{T}(\mathcal{D})\}$ on R under which $v(E)$ acts as E , and under which $u(F; v(E))$ acts as $\nabla_E F$.

Definition 6.1 *Suppose that U is a labeled ordered tree whose root has a single child and which acts on R as the differential operator E_U , and suppose that T is a labeled ordered tree which contains U as a subtree. Denote by $T(U|v(E_U))$ the labeled ordered tree resulting from replacing the subtree U with the tree $v(E_U)$. The action of $k\{\mathcal{T}(\mathcal{D})\}$ on R is called coherent if for all labeled ordered trees U whose root has a single child, and all labeled ordered trees T which contain U as a subtree, the actions on R of the trees T and $T(U|v(E_U))$ are identical, that is, $T \sim T(U|v(E_U))$.*

The actions defined in Example 3.2 and in Construction 3.7 are coherent.

Note that $k\{\mathcal{T}(\mathcal{D})\}$ is isomorphic as an algebra to the free associative algebra generated by the labeled ordered trees whose roots have only one child ([4][Th. 5.1] — what is called \mathcal{LOT} in [4] is called \mathcal{T} here) so non coherent actions of $k\{\mathcal{T}(\mathcal{D})\}$ on R can be easily constructed.

Theorem 6.2 *Let R be a commutative k -algebra, and let $\mathcal{D} = \text{Der}(R)$. Suppose a coherent action of $k\{\mathcal{T}(\mathcal{D})\}$ on R is given. Then the action of $k\{\mathcal{T}(\mathcal{D})\}$ on R is completely determined by the action E of the trees $v(E)$, and the action $\nabla_E F$ of the trees $u(F; v(E))$ for all $E, F \in \mathcal{D}$.*

PROOF: The proof uses two lemmas.

Lemma 6.3 *Let $E_1, \dots, E_n \in \mathcal{D}$. Then*

$$\begin{aligned} u(F; v(E_1), \dots, v(E_n)) &= \\ &v(E_1) \cdot u(F; v(E_2), \dots, v(E_n)) \\ &\quad - \sum_{i=2}^n u(F; v(E_2), \dots, u(E_i; v(E_1)), \dots, v(E_n)) \\ &\quad - t(v(E_1), u(F; v(E_2), \dots, v(E_n))). \end{aligned}$$

PROOF: The proof of this lemma follows immediately from the definition of multiplication for trees.

Lemma 6.4 *Suppose that $k\{\mathcal{T}(\mathcal{D})\}$ acts coherently on R . Let the action of the tree $u(F; v(E_2), \dots, v(E_n))$ on R be denoted by $G \in \mathcal{D}$, and let the action*

of the tree $u(E_i; v(E_1))$ on R be denoted by $H_i \in \mathcal{D}$. Then

$$\begin{aligned} u(F; v(E_1), \dots, v(E_n)) &\sim \\ &v(E_1) \cdot v(G) \\ &- \sum_{i=2}^n u(F; v(E_2), \dots, v(H_i), \dots, v(E_n)) - t(v(E_1), v(G)). \end{aligned}$$

PROOF: This lemma follows immediately from Lemma 6.3 and from the definition of coherence.

We now prove Theorem 6.2. The action of labeled trees with two nodes is determined by the action of \mathcal{D} on R . Repeated application of the definition of coherence shows that the action of trees with more than two nodes in $\mathcal{T}(\mathcal{D})$ is determined by the action of trees of the form $u(F; v(E_1), \dots, v(E_n))$. We prove by induction on n that this action is determined by the actions of E and $\nabla_F E$ for all $E, F \in \mathcal{D}$.

For $n = 1$ this is simply the assertion that the action of $u(E; v(F)) \sim \nabla_F E$ is determined. Suppose that the action is determined for n . We prove that it is determined for $n + 1$. Lemma 6.4 implies that the action of a tree of the form $u(F; v(E_1), \dots, v(E_{n+1}))$ is determined by the action of trees of the form $v(E)$, of trees of the form $u(F; v(E_1), \dots, v(E_n))$, and of trees of the form $t(v(E), v(F))$. The action of trees of the first form is given by hypothesis. The action of trees of the second form is determined by the induction hypothesis. The action of trees of the third form is determined by Lemma 3.5. This completes the proof of the theorem.

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